DIRECTORATE OF DISTANCE EDUCATION

UNIVERSITY OF NORTH BENGAL

MASTER OF SCIENCES- MATHEMATICS

SEMESTER -IV

INTEGRAL EQUATION AND INTEGRAL TRANSFORM

DEMATH4ELEC4

BLOCK-2

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FOREWORD

The Self Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

INTEGRAL EQUATION AND INTEGRAL TRANSFORM

BLOCK-1

BLOCK-2

BLOCK2 INTEGRAL EQUATION AND INTEGRAL TRANSFORM

Integral transform, mathematical operator that produces a new function $f(y)$ by integrating the product of an existing function $F(x)$ and a so-called kernel function $K(x, y)$ between suitable limits. The process, which is called transformation, is symbolized by the equation $f(y) = \int K(x, y)F(x)dx$. Several transforms are commonly named for the mathematicians who introduced them: in the Laplace transform, the kernel is $e^{-}xy$ and the limits of integration are zero and plus infinity; in the Fourier transform, the kernel is $(2\pi)^{-1/2}e^{-}ixy$ and the limits are minus and plus infinity.

Integral transforms are valuable for the simplification that they bring about, most often in dealing with differential equations subject to particular boundary conditions. Proper choice of the class of transformation usually makes it possible to convert not only the derivatives in an intractable differential equation but also the boundary values into terms of an algebraic equation that can be easily solved. The solution obtained is, of course, the transform of the solution of the original differential equation, and it is necessary to invert this transform to complete the operation. For the common transformations,

UNIT-8: HILBERT SPACE AND SYMMETRIC KERNELS

STRUCTURE

8.0 OBJECTIVES

Understand the concept of Symmetric Kernels

Comprehend the Complex Hilbert Space

Enumerate **Hilbert-Schmidt Theorem**

8.1 INTRODUCTION

In mathematical analysis, the **Hilbert–Schmidt theorem**, also known as the **eigenfunction expansion theorem**, is a fundamental result concerning compact, self-adjoint operators on Hilbert spaces. In the theory of partial differential equations, it is very useful in solving elliptic boundary value problems.

8.2 SYMMETRIC KERNELS

Symmetric Kernels: A kernel K(s, t) is symmetric (or complex symmetric or Hermitian) if

$$K(s, t) = K^*(t, s) \quad (8.1)$$

where the asterisk denotes the complex conjugate. If the kernel is real, then its symmetry is defined by the identity to the equality

$$K(s, t) = K(t, s) \quad (8.2)$$

An integral equation with a symmetric kernel is called a symmetric equation. We have seen in the previous chapters that the integral equations with symmetric kernels are of are of frequent occurrence in the formulation of physically motivated problems.

We claim that if a kernel is symmetric, then all its iterated kernels are also symmetric. Indeed

$$K_2(s,t) = \int K(s,x)K(x,t)\,dx = \int K^*(t,x)K^*(x,s)\,dx = K_2^*(t,s)$$

Again, if K_n (s, t) is symmetric, then the recursion relation gives

$$K_{n+1}(s,t) = \int K(s,x)K_n(x,t)\,dx \quad (8.3)$$

$$= \int K_n^*(t,x)K^*(x,s)\,dx = K_{n+1}^*(t,s)$$

The proof of our claim follows by induction. Note that the trace K(s, s) of a Symmetric kernel is always real because K(s, s) = K*(s, s). Similarly, the traces of all iterates are also real.

8.3 COMPLEX HILBERT SPACE

Notes

We present review of some important properties of the complex Hilbert space $\mathscr{L}_2$ (a, b) which is needed in the sequel. The same discussion will remain applicable to real space as a particular case. A linear space of infinite dimension with inner product (or scalar product) (x, y) which is a complex number is called a complex Hilbert space if it satisfies the following three axioms

(a) the definiteness axiom $(x, x) > 0$ for $x \neq 0$
(b) the linearity axiom $(\alpha x_1+\beta x_2, y) = \alpha(x_1, y) + \beta (x_2, y)$ where α and β are arbitrary complex numbers
(c) the axiom of (Hermitian) symmetry $(y, x) = (x, y^*)$
(d) Let *H* be the set of complex-valued functions ϕ(t) defined in the interval (a, b)such that

$$\int |\phi(t)|^2 \; dt < \infty \tag{8.4}$$

Furthermore, let us define the inner product by

$$(\phi, \psi) = \int \phi(t)\psi^*(t)\, dt \tag{8.5}$$

Then, H is a complex Hilbert space and is denoted as $\mathscr{L}_2$(a, b) or $\mathscr{L}_2$. The norm ‖ϕ‖ as defined by

$$\|\phi\| = \left(\int |\phi(t)|^2\right)^{\frac{1}{2}} \tag{8.6}$$

is called the norm that generates the natural metric

$$d(\phi, \psi) = \|\phi - \psi\| = (\phi - \psi, \phi - \psi)^{\frac{1}{2}} \tag{8.7}$$

A metric Space is called complete if every Cauchy sequence of functions in this space is a convergent sequence. A Hilbert space is an inner product linear space that is complete in its natural metric. The Schwarz and Minkowskii's inequalities as given by

The Schwarz and Minkowskii's inequalities as given by

$$|(\phi, \psi)| \leq \|\phi\|\|\psi\| \qquad (8.8)$$

$$|\phi + \psi| \leq \|\phi\| + \|\psi\| \qquad (8.9)$$

Incidentally, by a square-integrable function g(t), we mean that

$$\int_a^b |g(t)|^2 dt < \infty$$

A square-integrable function , *f(x)*is called an $\mathscr{L}_2$-function.

Another concept that is fundamental in the theory of Hilbert spaces is the concept of completeness. A metric space is called complete if every Cauchy sequence of functions in this space is a convergent sequence. If a metric space is not complete, then there is a simple way to add elements to this space to make it complete. A Hilbert space is an inner-product linear space that is complete in its natural metric. The completeness of $\mathscr{L}_2$ spaces plays an important role in the theory of linear operators such as the Fredholm operator K,

$$K\phi = \int K(s,t)\phi(t)dt \qquad (8.10)$$

The operator adjoint to is

$$K^*\psi = \int K^*(t,s)\psi(t)dt \qquad (8.11)$$

The operators (8.10) and (8.11) are connected by the interesting relationship

$$(K\phi, \psi) = (\phi, K^* \psi) \qquad (8.12)$$

which proved as follows

$$(K\phi,\psi)= \int \psi^*(s)\left[\int K(s,t)\phi(t)dt\right]ds$$

$$= \int \phi(t)\left[\int K(s,t)\psi^*(s)ds\right]dt$$

$$= \int \phi(s)\left[\int K(t,s)\psi^*(t)dt\right]ds$$

$$= \int \phi(s)\left[\int K^*(t,s)\psi(t)dt\right]^* ds$$

$$= (\phi, K^*\psi)$$

For a symmetric kernel, this result becomes

$$(K\phi,\psi) = (\phi, K^*\psi) \qquad (8.13)$$

that is, a symmetric operator is self-adjoint. Note that permutation of factors in a scalar product is equivalent to taking the complex conjugate; that is, $(\phi, K\phi) = (K\phi, \phi)^*$. Combining this with Equation (8.13), we find that, for a symmetric kernel, the inner product $(K\phi, \phi)$ is always real; the converse is also true.

8.3.1 An Orthonormal System of Functions

Systems of orthogonal functions play an important role in the theory of integral equation and their applications. A finite or an infinite set $\{\phi_k\}$ is said to be an orthogonal set if

$$(\phi_i, \phi_j) = 0, i \neq j. \quad or \int_a^b \phi_i(x)\phi_j(x)dx = 0\,, \quad i \neq j$$

If none of the elements of this set is a zero vector, then it is said to be a proper orthogonal set. The set $\{\phi i(x)\}$is orthonormal if

$$(\phi_i, \phi_j) = \int_a^b \phi_i(x)\phi_j(x)dx = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Any function $\phi(x)$ for which $\|\phi\| = 1$ are said to be normalized.

Given a finite or an infinite (denumerable) independent set of functions $\{\psi_1,\psi_2,\ldots,\psi_k,..\}$ we can construct an orthonormal set $\{\phi_1,\phi_2,\ldots,\phi_k,..\}$ by the well· known Gram-Schmidt procedure as follows.

Let $\phi_1 = \frac{\psi_1}{||\psi_1||}$

To obtain ϕ_2, we define

$$\omega_2(s) = \psi_2(s) - (\psi_2, \phi_1)\phi_1$$

The function ω_2 is orthogonal to ϕ_1. Hence ϕ_2 can be constructed by setting $\phi_2 = \frac{\omega_2}{||\omega_2||}$. Proceeding in this manner, we obtain

$$\omega_k(s) = \psi_k(s) - \sum_{1}^{k-1}(\psi_k, \phi_i)\phi_i\ , \qquad \phi_k = \frac{\omega_k}{\|\omega_k\|}. \tag{8.13}$$

We have, thereby, obtained an equally numerous set of orthonormal functions. Again, if we are given a set of orthogonal functions, we can convert it into an orthonormal set simply by dividing each function by its norm. Starting from an arbitrary orthonormal system, it is possible to construct the theory of Fourier series. Suppose we want to find the best approximation of an arbitrary function $\psi(x)$ in terms of a linear combination of an orthonormal set $\{\phi_1,\phi_2,\ldots,\phi_n\}$. By the best approximation, we mean that we choose the coefficients $\alpha_1,\alpha_2,\ldots,\alpha_n$ such as to minimize

$$\|\psi - \textstyle\sum_{i=1}^{n} \alpha_i\phi_i\|.$$

or, what is equivalent, to minimize

$$\|\psi - \textstyle\sum_{i=1}^{n} \alpha_i\phi_i\|^2$$

Now, for any $\alpha_1, \alpha_2, \ldots, \alpha_n$ we have

$$\left\|\psi - \sum_{i=1}^{n} \alpha_i \phi_i\right\|^2 = \|\psi\|^2 + \sum_{i=1}^{n} |(\psi, \phi_i) - \alpha_i|^2 - \sum_{i=1}^{n} |(\psi, \phi_i)|^2 \qquad (8.14)$$

It is obvious that the minimum is achieved by choosing $\alpha_i = (\psi, \phi_i) = \alpha_i$ (say) The numbers a_i are called the Fourier coefficients of the function ψ (s) relative to the orthonormal system $\{\phi_i\}$. In that case, the relation (8.14) reduces to

$$\left\|\psi - \sum_{i=1}^{n} \alpha_i \phi_i\right\|^2 = \|\psi\|^2 - \sum_{i=1}^{n} |a_i|^2 \qquad (8.15)$$

Since the quantity on the L.H.S. of (8.15), we obtain

$$\sum_{i=1}^{n} |a_i|^2 \leq \|\psi\|^2 \qquad (8.16)$$

which, for the infinite set $\{\phi_i\}$ leads to the Bessel inequality

$$\sum_{i=1}^{\infty} |a_i|^2 \leq \|\psi\|^2 \qquad (8.17)$$

Assuming that we are given an infinite orthonormal system $\{\phi_i(s)\}$ in $\mathscr{L}$, and a sequence of constants $\{\alpha_i\}$; then the, convergence of the series $\sum_{k=1}^{\infty} |\alpha_k^2|$ is clearly a necessary condition for the existence of $\mathscr{L}_2$-function ,*f(s)* whose Fourier coefficients with respect to the system ϕ_i are α_i. It so happens that this condition is also sufficient and the result is embodied in the Riesz-Fischer theorem, which we state as follows without proof.

8.3.2 Riesz-Fischer Theorem

If { ϕ_i (s)} is a given orthonormal system of functions in $\mathscr{L}_2$ and { α_i }is a given sequence of complex numbers such that the series $\sum_{k=1}^{\infty} |\alpha_k^2|$ converges, then there exists a unique function ,*f(s)* for which α_i are the Fourier coefficients with respect to the orthonormal system { ϕ_i }and to which the Fourier series converges in the mean, that is

$$\left\| f(s) - \sum_{i=1}^{n} \alpha_i \phi_i \right\| \to 0 \quad \text{as} \quad n \to \infty.$$

If an orthonormal system of functions 3 can be found in $\mathscr{L}$-space such that every other element of this space can be represented linearly in terms of this system, then it is called an orthonormal basis. The concepts of an orthonormal basis and a complete system of orthonormal functions are equivalent. Indeed, if any of the following criteria are met, the orthonormal set {$\phi_1,\phi_2,\ldots,\phi_n$} is complete.

a) For every function ψ in $\mathscr{L}_2$,

$$\psi = \sum_i (\psi, \phi_i)\phi_i = \sum_i a_i \phi_i \qquad (8.18)$$

b) For every function in $\mathscr{L}_2$,

$$\|\psi\|^2 = \sum_{i=1}^{\infty} |(\psi, \phi_i)|^2 \qquad (8.19)$$

This is called Parseval's identity.

c) The only function ψ in $\mathscr{L}_2$ for which all the Fourier coefficients vanish is the trivial function (zero function).

d) There exists no function ψ in $\mathscr{L}_2$ such that $\{\psi, \phi 1, \ldots \phi_k, \ldots\}$ is an orthonormal set.

The equivalence of these different criteria can be easily proved. One frequently encounters Fourier series of somewhat more general character. Let *r(t)* be a continuous, real, and non-negative function in the interval (a, b). We say that the set functions $\{\phi_i\}$ is orthonormal with weight *r(t)* if

$$\int r(t)\phi_j(t)\phi_k^*(t)dt = \begin{cases} 0\,, & j \neq k, \\ 1\,, & j = k \end{cases} \tag{8.20}$$

The Fourier expansions in terms of such functions are treated by introducing a new inner product with the corresponding norm

$$(\phi, \psi) = \int r(t)\phi_j(t)\psi^*(t)dt \tag{8.21}$$

$$\|\phi\|_r = \left[\int r(t)\phi_j(t)\phi_k^*(t)dt\right]^{\frac{1}{2}} \tag{8.22}$$

The space of functions for which $\|\phi\|_r < \infty$ is a Hilbert space and all the preceding results hold. Some examples of the complete orthogonal and orthonormal systems are listed in the following.

a) The system

$$\phi_k(s) = (2\pi)^{-\frac{1}{2}} e^{iks}$$

is orthonormal, where k is any integer $-\infty < k < \infty$.

b) The Legendre polynomials

$$P_0(s) = 1\,, \quad P_n(s) = \frac{1}{2^n n!}\frac{d^n(s^2-1)^n}{ds^n}\,, \qquad n = 1,2,\cdots$$

Are orthogonal in the interval (−1,1). Indeed,

$$\int_{-1}^{1} P_j(s)P_k(s)ds = \begin{cases} 0, & j \neq k \\ \dfrac{2}{2k+1} & j = k \end{cases} \qquad (8.23)$$

c) Let $\alpha_{k,n}$, denote the positive zeros of the Bessel function Jn(s), k = 1,2, ⋯ , n > −1. The system of functions $J_n(\alpha_{k\,n}\, s)$ is orthogonal with weight r(s) = s in the interval (0,1):

$$\int_{0}^{1} sJ_n(\alpha_{j,n}s)J_n(\alpha_{k,n}s)ds = \begin{cases} 0, & j \neq k \\ J_{n+1}^2(\alpha_{k,n}) & j = k \end{cases} \qquad (8.24)$$

Check your Progress-1

1. Define Orthogonal set

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2. State the criteria for orthonormal set to be complete

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8.4 HILBERT-SCHMIDT THEOREM

The pivotal result in the theory of symmetric integral equations is embodied in the following theorem

$$f(s) = \sum_{n=1}^{\infty} f_n \phi_n(s), \quad f_n = (f, \phi_n).$$

Hilbert-Schmidt Theorem: If f(s) can be written in the form

$$f(s) = \int K(s,t)h(t)\,dt = 0 \qquad (8.25)$$

where K(s, t) is a symmetric $\mathscr{L}_2$-kernel and *h*(t) is an $\mathscr{L}$-function, then *f(s)* can be expanded in an absolutely and uniformly convergent Fourier series with respect to the orthonormal system of eigenfunctions of the kernel *K* .The Fourier coefficients of the function ,f(s) are related to the Fourier coefficients h_n of the function *h*(*s*) by the relations where λ_n are the eigenvalues of the kernel .

Proof. The Fourier coefficients of the function ,*f(s)* with respect to the orthonormal system $\{\phi_n(s)\}$

$$f_n = (f,\phi_n) = (Kh,\phi_n) = (h,K\phi_n) = \lambda_n^{-1}(h,\phi_n) = \lambda_n^{-1}h_n$$

where we have used the self-adjoint property of the operator as well as the relation $\lambda_n K\phi_n = \phi_n$. Thus, the Fourier series for *f(s)* is

$$f(s) \sim \sum_{n=1}^{\infty} f_n\phi_n(s) = \sum_{n}^{\infty} \frac{h_n}{\lambda_n}\phi_n(s) \qquad (8.27)$$

The remainder term for this series can be estimated as follows :

$$\left|\sum_{k=n+1}^{n+p} h_k \frac{\phi_k(s)}{\lambda_k}\right|^2 \leq \sum_{k=n+1}^{n+p} h_k^2 \sum_{k=n+1}^{n+p} |\phi_k(s)|^2/\lambda_k^2$$

$$\leq \sum_{k=n+1}^{n+p} h_k^2 \sum_{k=1}^{\infty} \frac{|\phi_k(s)|^2}{\lambda_k^2} \qquad (8.28)$$

We find that the above series is bounded. Also, because *h*(s) is a $\mathscr{L}_2$-function, the series $\sum_{k=1}^{\infty} h_k^2$ s convergent and the partial sum $\sum_{k=n+1}^{n+p} h_k^2$. can be made arbitrarily small. Therefore, the series (8.27) converges absolutely and uniformly. It remains to be shown that the series (8.27) converges to *f(s)* in the mean. To this end, let us denote its partial sum as

$$\psi_n(s) = \sum_{m=1}^{\infty} \frac{h_m}{\lambda_m} \phi_m(s) \qquad (8.29)$$

and estimate the value of $\|f(s) - \psi(s)\|$. Now,

where K_{n+1} is the truncated kernel as defined in the previous section. From (8.30), we obtain

$$f(s) - \psi_n(s) = Kh - \sum_{m=1}^{\infty} \frac{h_m}{\lambda_m} \phi_m(s)$$

$$= Kh - \sum_{m=1}^{\infty} \frac{(h, \phi_m)}{\lambda_m} \phi_m(s) = K^{n+1}h \qquad (8.30)$$

$$\|f(s) - \psi_n(s)\|^2 = \|K^{n+1}h\|^2 = (K^{n+1}h, K^{n+1}h)$$

$$= (h, K^{n+1}K^{n+1}h) = (h, K_2^{n+1}h) \qquad (8.31)$$

where we have used the self-adjointness of the kernel K^{n+1} and also the relation $K^{n+1} = K_h^{n+1}$. We find that the least eigenvalue of the kernel K_2^{n+1} is equal to λ_{n+1}^2.

$$\frac{1}{\lambda_{n+1}^2} = max\left[\frac{(h, K_2^{n+1}h)}{h, h}\right] \qquad (8.32)$$

where we have omitted the modulus sign from the scalar product (h, $K_2^{n+1}h$), as it is a positive quantity. Combining (8.31) and (8.32), we have

$$\|f(s) - \psi_n(s)\|^2 = (h, K_2^{n+1}h) \leq \frac{(h, h)}{\lambda_{n+1}^2}$$

Since $\lambda_{n+1} \rightarrow \infty$, we find that $\|f(s) - \psi_n(s)\| \rightarrow 0$ *as* $n \rightarrow \infty$.

Finally, we use the relation

$$\|f-\psi_n\| \le \|f-\psi_n\| + \|\psi_n-\psi\| \qquad (8.33)$$

Where is the limit of the series with partial sum , to prove that , f = ψ .
The first term on the right side of (8.33) tends to zero, as proved above. To prove that the second term also tends to zero, we observe that, since the series (8.27) converges uniformly, we have, for an arbitrarily small and positive ϵ,

$$|\psi_n(s)-\psi(s)| < \epsilon,$$

When L is sufficiently large. Hence, and the result follows.

$$,\|\psi_n(s)-\psi(s)\| < \epsilon(b-a)^{\frac{1}{2}}$$

Remark. It is to be noted that we assumed neither the convergence of the Fourier series *h*(s) nor the completeness of the orthonormal system. We have merely used the fact that h is an $\mathscr{L}_2$-function. An immediate consequence of the Hilbert-Schmidt theorem is the bilinear form. Indeed,

$$K_m(s,t) = \int K(s,x)K_{m-1}(x,t)dx\,, \qquad m = 2,3,\cdots \qquad (8.34)$$

by definition,

which is of the form (8.25) with h(s) = K_{m-1} (s, t) ; t fixed. The Fourier coefficient a_k(t) of K_m (s, t) with respect to the system of eigen functions {ϕ_k(s)} of K (s, t) is

$$a_k(t) = \int K_m(s,t)\phi_k^*(s)ds = \lambda_k^{-m}\phi_k^*(t)$$

It follows from the above theorem that all the iterated kernels K_m (s, t), m ≥ 2, of a symmetric $\mathscr{L}$-kernel can be represented by the absolutely and uniformly convergent series

$$K_m(s,t) = \sum_{k=1}^{\infty} \lambda_k^{-m}\phi_k(s)\phi_k^*(t) \qquad (8.35)$$

By setting s = t in (8.35) and integrating from a to b, we obtain

$$\sum_{k=1}^{\infty} \lambda_k^{-m} = \int K_m(s,s)ds = A_m \qquad (8.36)$$

where Am is the trace of the iterated kernel Km. Next, we apply the Riesz-Fischer theorem and find from (8.37) with m = 2 that the series

$$\sum_{k=1}^{\infty} \frac{\phi_k(s)\phi_k^*(t)}{\lambda_k} \qquad (8.37)$$

converges in the mean to a symmetric $\mathscr{L}_2$-kernel K(s, t) which, considered as a Fredholm kernel, has precisely the sequence of numbers $\{\lambda_k\}$ as eigen values.

Check your Progress-2

3. State **Hilbert-Schmidt Theorem**

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8.5 LET US SUM UP

We have discussed about symmetric kernel and then all its iterated kernels are also symmetric. We discussed about Hilbert and Metric space. Systems of orthogonal functions play an important role in the theory of integral equation and their applications. We discussed about Riesz-Fischer Theorem & Hilbert-Schmidt Theorem

8.6 KEYWORDS

Notes

Self- Adjoint: In functional analysis, a linear operator A on a Hilbert space is called **self-adjoint** if it is equal to its own **adjoint** A* and that the domain of A is the same as that of A*

Partial Sum - A **Partial Sum** is the sum of part of the sequence

Scalar Product: a scalar function of two vectors, equal to the product of their magnitudes and the cosine of the angle between them

8.7 QUESTIONS FOR REVIEW

1. Using Hilbert-Schmidt theorem, solve the following symmetric integral equations

(i) $y(x) = x + \int_0^1 (x+1)y(t)dt \ , \lambda \neq \lambda_1, \lambda_2$

(ii) $y(x) = (1 - x\sqrt{3}) + (-6 + 4\sqrt{3}) \int_0^1 (x+t)y(t)dt.$

2. Prove Hilbert-Schmidt theorem

8.8 SUGGESTED READINGS AND REFERENCES

1. M. Gelfand and S. V. Fomin. Calculus of Variations, Prentice Hall.
2. Linear Integral Equation: W.V. Lovitt (Dover).
3. Integral Equations, Porter and Stirling, Cambridge.
4. The Use of Integral Transform, I.n. Sneddon, Tata-McGrawHill, 1974
5. R. Churchil& J. Brown Fourier Series and Boundary Value Problems, McGraw-Hill, 1978
6. D. Powers, Boundary Value Problems Academic Press, 1979.

8.9ANSWERS TO CHECK YOUR PROGRESS

Notes

1. Provide definition – Refer 8.3.2

2. Provide 4 criteria – 8.3.2

3. Provide statement with equation – 8.4

UNIT 9: ABEL'S INTEGRAL EQUATION

STRUCTURE

9.0 OBJECTIVES

Understand the Singular integral equation & its Inversion Formula

Enumerate Abel's Problem & The Generalized Abel's Integral Equation

9.1 INTRODUCTION

Niels Henrik Abel devised what is now known as Abel's Integral Equation as a tool by which to solve the Tautochrone Problem in 1823. Abel in 1823 investigated the motion of a particle that slides down along a smooth unknown curve, in a vertical plane, under the influence of the gravitational field. The main goal of Abel's problem is to determine the unknown function *g(x)* under the integral sign that will define the equation of the curve.

9.2 SINGULAR INTEGRAL EQUATION

An integral equation is called a singular integral equation if one or both limits of integration becomes infinite, or if the kernel K(s, t) of the equation becomes infinite at one or more points in the interval of integration. To be specific, the integral equation of the first kind

$$f(x) = \lambda \int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)dt \qquad (9.1)$$

Or the integral equation of second kind

$$u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)dt \qquad (9.2)$$

is called singular if α (x), or β (x) or both limits of integration are infinite. Moreover, the equation (9.1) or (9.2) is also called a singular integral equation if the kernel K (s, t) becomes infinite at one or more points in the domain of integration. Examples of the first style of singular integral equations are given by the following examples

$$u(x) = e^x + \int_0^\infty K(x,t)u(t)dt \qquad (9.3)$$

$$F\{u(x)\} = \int_{-\infty}^{\infty} e^{-iwx}u(x)dx \qquad (9.4)$$

$$L[u(x)] = \int_0^\infty e^{-sx}u(x)dx \qquad (9.5)$$

The integral equations (9.4) and (9.5) are Fourier transform and Laplace transform of the function *u(x)* respectively. In fact these two equations are Fredholm integral equations of the first kind with kernels given by

K(x, w) = e $^{-iwx}$ *and* K(x, s) = e $^{-sx}$. Equations (9.3)-(9.5)can be defined also as the improper integrals because of the limits of integration are infinite.

Examples of the second type of singular integral equations are given by the following:

$$f(x) = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt \qquad (9.6)$$

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt \qquad (9.7)$$

$$u(x) = f(x) + \lambda \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt \qquad (9.8)$$

where the singular behavior in these examples is attributed to the kernel K(x, t) becoming infinite as x → ∞.

Remark It is important to note that integral equations (9.6) and (9.7) are called Abel's problems and generalized Abel's integral equations, respectively, after the name of the Norwegian mathematician Niels Abel who invented them in 1823 in his research of mathematical physics. Singular equations (9.8) are called the weakly-singular second kind Volterra type integral equations.

9.3 ABEL'S PROBLEM

Abel in 1823 investigated the motion of a particle that slides down along a smooth unknown curve, in a vertical plane, under the influence of the gravitational field. It is assumed that the particle starts from rest at a point *P*, with vertical elevation *x*, slides along the unknown curve, to the lowest point *O* on the curve where the vertical distance is $x = 0$. The total time of descent from the highest point to the lowest point on the curve is given in advance, and dependent on the elevation *x*, hence expressed by

$$T = h(x) \qquad (9.9)$$

Assuming that the curve between the points P and O has an arc lengths, then the velocity at a point Q on the curve, between P and O, is given by

$$\frac{ds}{dT} = -\sqrt{2g(x-t)} \qquad (9.10)$$

Where t is a variable coordinate defines the vertical distance of the point Q, and g is a constant defines the acceleration of gravity. Integrating both sides of (9.10) gives

$$T = \int_{O}^{P} \frac{ds}{\sqrt{2g(x-t)}} \qquad (9.11)$$

Setting

$$ds = g(t)dt \qquad (9.12)$$

and using (9.9), we find that the equation of motion of the sliding particle is governed by

$$f(x) = \int_{0}^{x} \frac{1}{\sqrt{x-t}} g(t)dt \qquad (9.13)$$

We point out that $f(x)$ is a pre-determined function that depends on the elevation and given by

$$f(x) = \sqrt{2g}h(x) \qquad (9.14)$$

where g is the gravitational constant, and $h(x)$ is the time of descent from the highest point to the lowest point on the curve. The main goal of Abel's problem is to determine the unknown function $g(x)$ under the integral sign that will define the equation of the curve. Having determined $g(x)$, the equation of the smooth curve, where the particle slides along, can be easily obtained using the calculus formulas related to the arc length concepts.

It is worth mentioning that Abel's integral equation (9.13) is also called Volterra integral equation of the first kind. Besides, the kernel K (s, t) in (9.13) is given by

$$K(x,t) = \frac{1}{\sqrt{x-t}} \qquad (9.14)$$

which shows that the kernel (9.14) is singular in that

K (s, t) $\rightarrow \infty$ as t $\rightarrow x$ (9.15)

The interesting Abel's problem has been approached by different methods. In the following we will employ Laplace transforms only to determine a suitable formula to solve Abel's problem (9.13), noting that Laplace transforms will not be used in our approach to handle the singular equations. Taking Laplace transforms of both sides of (9.13) leads to

$$L[f(x)] = L[g(x)]L\left[x^{-\frac{1}{2}}\right]$$

$$= L[g(x)]\frac{\Gamma\left(\frac{1}{2}\right)}{z^{\frac{1}{2}}} \qquad (9.16)$$

where Γ is the gamma function. Noting that Γ (1/2) = $\sqrt{\pi}$ the equation (9.16) becomes

$$L[g(x)] = \frac{z^{\frac{1}{2}}}{\sqrt{\pi}}L[f(x)] \qquad (9.17)$$

which can be rewritten by

$$L[g(x)] = \frac{z}{\pi}\left(\sqrt{\pi}z^{-\frac{1}{2}}L[f(x)]\right) \qquad (9.18)$$

Setting

$$h(x) = \int_0^x (x-t)^{-\frac{1}{2}} f(t)dt \qquad (9.19)$$

Into (9.18) yields

$$L[g(x)] = \frac{z}{\pi}(L[h(x)]) \qquad (9.20)$$

which gives

$$L[g(x)] = \frac{1}{\pi} L[h'(x)] \qquad (9.21)$$

upon using the fact

$$L[h'(x)] = zL[h(x)] \qquad (9.22)$$

Applying L^{-1} to both sides of (9.21) yields the easily calculable formula

$$g(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{\sqrt{x-t}}dt \qquad (9.23)$$

that will be used for the determination of the solution. It is clear that Leibniz rule is not applicable in (9.23) because the integrand is discontinuous at the interval of integration. As indicated earlier, determination of *g(x)* will lead to the determination of the curve where the particle slides along this curve.

It is obvious that Abel's problem given by (9.13) can be solved now by using the formula (9.23) where the unknown function *g(x)* has been replaced by the given function *f(x)*. One last remark concerns the use of the formula (9.23). The process consists of selecting the proper substitution for ($x - t$), integrate the resulting definite integral and finally differentiate the result of the evaluation.

The procedure of using the formula (9.23) that determines the solution of Abel's problem (9.13) will be illustrated by the following examples.

Example 1. As a first example we consider the following Abel's problem

$$s = \int_0^s \frac{1}{\sqrt{s-t}} g(t)dt \qquad (9.24)$$

Substituting ,*f(s)* = *s* in (9.23) yields

$$g(t) = \frac{1}{\pi}\frac{d}{dt}\int_0^t \frac{s}{\sqrt{t-s}} ds$$

$$= \frac{1}{\pi}\frac{d}{dt}\left[-\frac{2}{3}(s+2t)\sqrt{t-s}\right]_0^t \qquad (9.25)$$

$$= \frac{1}{\pi}\frac{d}{dt}\left[\frac{4}{3}t^{\frac{3}{2}}\right] = \frac{2t^{\frac{1}{2}}}{\pi} \qquad (9.26)$$

Example 2. Solve the following Abel's problem

$$\pi/2\, x = \int_0^x \frac{1}{\sqrt{x-t}}\, g(t)dt \qquad (9.27)$$

Substituting, *f(x)* = $\frac{\pi}{2}x$ in (9.23) gives

$$g(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\frac{\pi}{2}t}{\sqrt{x-t}} dt$$

$$= \frac{1}{2}\frac{d}{dx}\int_0^x \frac{1}{\sqrt{x-t}} dt \qquad (9.28)$$

Using integration by substitution, where we set *y* = *x* − *t*, we obtain

$$g(x) = \frac{1}{2}\frac{d}{dx}\left(\frac{4}{3}x^{\frac{3}{2}}\right)$$

$$= x^{\frac{1}{2}} \qquad (9.29)$$

Check your Progress-1

1. Define Singular integral equation

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2. Discuss Abel's Problem

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9.4 THE GENERALIZED ABEL'S INTEGRAL EQUATION

It is important here to note that Abel introduced the more general singular integral equation

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha}\, g(t)dt, \qquad 0 < \alpha < 1, \qquad (9.30)$$

known as the Generalized Abel's integral equation, where the exponent of the denominator of the kernel is α, such as 0 < α < 1. It can be easily seen that Abel's problem discussed above is a special case of the generalized equation where α = 1/2. To determine a practical formula for the solution *g(x)* of (9.30), and hence for the Abel's problem, we simply use the Laplace transform in a similar manner to that used above. As noted before, the Laplace transform will be used for the derivation of the proper formula, but will not be used in handling the equations. Taking Laplace transforms to both sides (9.30) yields,

$$\begin{aligned} L[f(x)] &= L[g(x)]L[x^{-\alpha}] \\ &= L[g(x)]\frac{\Gamma(1-\alpha)}{z^{1-\alpha}} \qquad (9.31) \end{aligned}$$

Notes

Where Γ is the gamma function. The equation (9.31) can be written as

$$L[g(x)] = \frac{z}{\Gamma(\alpha)\Gamma(1-\alpha)}\Gamma(\alpha)z^{-\alpha}L[f(x)] \tag{9.32}$$

Or equivalently

$$L[g(x)] = \frac{z}{\Gamma(\alpha)\Gamma(1-\alpha)}L[u(x)] \tag{9.33}$$

Where

$$g(x) = \int_0^x (x-t)^{\alpha-1}f(t)dt \tag{9.34}$$

using (9.34) into (9.33) yields upon using the identities

$$L[u'(x)] = zL[u(x)] \tag{9.36}$$

And

$$L[g(x)] = \frac{\sin\alpha\pi}{\pi}L[u'(x)] \tag{9.35}$$

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin\alpha\pi} \tag{9.37}$$

Applying L^{-1} to both sides of (9.35) yields the easily computable formula for determining the solution.

$$g(x) = \frac{\sin\alpha\pi}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt\,, \qquad 0<\alpha<1 \tag{9.38}$$

Recall that, *f(x)* is differentiable, therefore we can derive a more suitable formula that will support our computational of (9.38) by parts where we obtain

$$\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt = -\frac{1}{\alpha}[f(t)(x-t)^{\alpha}]_0^x + \frac{1}{\alpha}\int_0^x (x-t)^{\alpha} f'(t)dt \qquad (9.39)$$

$$= \frac{1}{\alpha} f(0)x^{\alpha} + \frac{1}{\alpha}\int_0^x (x-t)^{\alpha} f'(t)dt$$

Differentiating both sides of (9.39), noting that Leibniz rule should be used in differentiating the integral at the right hand side, yields

$$\frac{d}{dx}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt = \frac{f(0)}{(x)^{1-\alpha}} + \int_0^x \frac{f'(t)}{(x-t)^{1-\alpha}}dt \qquad (9.40)$$

Substituting (9.40) into (9.38) yields the desired formula given by

$$g(x) = \frac{\sin \alpha\pi}{\pi}\left(\frac{f(0)}{x^{1-\alpha}} + \int_0^x \frac{f'(t)}{(x-t)^{1-\alpha}}dt\right), \quad 0 < \alpha < 1, \qquad (9.41)$$

That will be used to determine the solution of the generalized Abel's equation and consequently, of the standard Abel's problem as well. This will be illustrated by examining the following examples.

Example 1. Solve the following generalized Abel's integral equation

$$27x^{\frac{8}{3}} = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} g(t)dt. \qquad (9.42)$$

Notice that

$$\alpha = \frac{1}{3}, f(x) = 27\, x^{\frac{8}{3}}.$$

Using (9.38) gives

$$g(x) = \frac{\sqrt{3}}{2\pi}\frac{d}{dx}\int_0^x \frac{27t^{\frac{8}{3}}}{(x-t)^{\frac{2}{3}}}dt = 40x^2 \qquad (9.43)$$

Example 2. Solve the following generalized Abel's integral equation

$$4\sqrt{3}\pi x = \int_0^x \frac{1}{(x-t)^{\frac{2}{3}}}g(t)dt \qquad (9.44)$$

Notice that

$$\alpha = \frac{2}{3}, f(x) = 4\sqrt{3}\pi x.$$

Using (9.38) gives

$$g(x) = \frac{\sqrt{3}}{2\pi}\frac{d}{dx}\int_0^x \frac{4\sqrt{3}\pi t}{(x-t)^{\frac{1}{3}}}dt = 9x^{\frac{2}{3}} \qquad (9.45)$$

Example 3. Solve the following generalized Abel's integral equation

$$\frac{9}{10}x^{\frac{5}{3}} = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}}g(t)dt. \qquad (9.46)$$

Notice that

$$\alpha = \frac{1}{3}, f(x) = \frac{9}{10}x^{\frac{5}{3}}.$$

Using (9.38) gives

$$g(x) = \frac{\sqrt{3}}{2\pi}\frac{d}{dx}\int_0^x \frac{4\frac{9}{10}t^{\frac{5}{3}}}{(x-t)^{\frac{2}{3}}}dt = x \qquad (9.47)$$

9.5 INVERSION FORMULA FOR SINGULAR INTEGRAL EQUATION

The integral equation (9.30) is a special case of the singular integral equation

$$f(s) = \int_a^s \frac{g(t)dt}{[h(s) - h(t)]^\alpha}, \quad 0 < \alpha < 1 \tag{9.48}$$

Where *h*(t) is a strictly monotonically increasing and differentiable function in (a, b) and *h*'(t) ≠ 0 in this interval. To solve this, we consider the integral

$$\int_a^s \frac{h'(u)f(u)du}{[h(s) - h(u)]^{1-\alpha}}$$

And substitute for f(u)from (9.48). This gives

$$\int_a^s \frac{h'(u)g(t)dtdu}{[h(u) - h(t)]^\alpha [h(s) - h(u)]^{1-\alpha}}$$

which, by change of the order of integration, becomes

$$\int_a^s g(t)dt \int_t^s \frac{h'(u)du}{[h(u) - h(t)]^\alpha [h(s) - h(u)]^{1-\alpha}}$$

The inner integral is easily proved to be equal to the beta function B(α, 1 — α). We have thus proved that

$$\int_a^s \frac{h'(u)f(u)du}{[h(s) - h(u)]^{1-\alpha}} = \frac{\pi}{\sin \alpha\pi} \int_a^s g(t)dt \tag{9.49}$$

and by differentiating both sides of (9.49), we obtain the solution

$$g(t) = \frac{\sin \alpha\pi}{\pi} \frac{d}{dt} \int_a^t \frac{h'(u) f(u) du}{[h(s) - h(u)]^{1-\alpha}} \quad (9.50)$$

Similarly, the integral equation

$$f(s) = \int_s^b \frac{g(t)dt}{[h(s) - h(u)]^{\alpha}}, \quad 0 < \alpha < 1 , \quad (9.51)$$

and a < s < b, with h(t) a monotonically increasing function, has the solution

$$g\ (t) = -\frac{\sin \alpha\pi}{\pi} \frac{d}{dt} \int_s^b \frac{h'(u) f(u) du}{[h(s) - h(u)]^{1-\alpha}} \quad (9.52)$$

We close this section with the remark that a Fredholm integral equation with a kernel of the type

$$K(s,t) = \frac{H(s,t)}{(t-s)^{\alpha}}, \quad 0 < \alpha < 1 , \quad (9.53)$$

Where $H(s, t)$ is a bounded function, can be transformed to a kernel which is bounded. It is done by the method of iterated kernels. Indeed, it can be shown that, if the singular kernel has the form as given by the relation (9.53), then there always exists a positive integer p_0, dependent on α, such that, for $p > p_0$ the iterated kernel Kp(s, t) is bounded. For this reason, the kernel (9.53) is called weakly singular.

Note that, for this hypothesis, the condition α < 1 is essential. For the important case α = 1, the integral equation differs radically from the equations considered in this section. Moreover, we need the notion of Cauchy principal value for this case. But, before considering the case α = 1, let us give some examples for the case α < 1.

Example 1. Solve the integral equation

$$f(s) = \int_a^s \frac{g(t)dt}{(\cos t - \cos s)^{\frac{1}{2}}}, \quad 0 \leq a < s < b \leq \pi \tag{9.54}$$

Comparing this with integral equation (9.48), we see that α = 1/2 and *h*(t) = 1 − cos t, a strictly monotonically increasing function in (0, *π*). Substituting these values for *h*(*u*) in (9.50), we have the required solution

$$g(t) = \frac{1}{\pi}\frac{d}{dt}\left[\int_a^t \frac{\sin u\, f(u)du}{(\cos u - \cos t)^{\frac{1}{2}}}\right], \quad a < t < b \tag{9.55}$$

Similarly, the integral equation

$$f(s) = \int_s^b \frac{g(t)dt}{(\cos s - \cos t)^{\frac{1}{2}}}, \quad 0 \leq a < s < b \leq \pi \tag{9.56}$$

has the solution

$$g(t) = -\frac{1}{\pi}\frac{d}{dt}\left[\int_t^b \frac{\sin u\, f(u)du}{(\cos u - \cos t)^{\frac{1}{2}}}\right], \quad a < t < b \tag{9.57}$$

Example 2. Solve the integral equation

(a) $$f(s) = \int_a^s \frac{g(t)dt}{(s^2 - t^2)^{\alpha}}, \quad 0 < \alpha < 1;\ a < s < b \tag{9.58}$$

And

(b) $$f(s) = \int_s^b \frac{g(t)dt}{(t^2 - s^2)^{s}}, \quad 0 < \alpha < 1;\ a < s < b \tag{9.59}$$

Using (9.48) and (9.58), we find that $h(t) = t^2$, which is a strictly monotonic function. The solution, therefore, follows from (9.50)

Notes

$$g(t) = \frac{2\sin\alpha\pi}{\pi}\frac{d}{dt}\int_a^t \frac{uf(u)du}{[t^2-u^2]^{1-\alpha}}, \quad a<t<b \tag{9.60}$$

Similarly, the solution of the integral equation (9.59) is

$$g(t) = -\frac{2\sin\alpha\pi}{\pi}\frac{d}{dt}\int_t^b \frac{uf(u)du}{[u^2-t^2]^{1-\alpha}}, \quad a<t<b \tag{9.61}$$

The results (9.60) and (9.61) remain valid when a tends to 0 and b tends to +∞. Hence, the solution of the integral equation

$$f(s) = \int_0^s \frac{g(t)dt}{(s^2-t^2)^\alpha}, \quad 0<\alpha<1, \tag{9.62}$$

$$g(t) = \frac{2\sin\alpha\pi}{\pi}\frac{d}{dt}\int_0^t \frac{uf(u)du}{[u^2-t^2]^{1-\alpha}} \tag{9.63}$$

Similarly, the solution of the integral equation

$$f(s) = \int_s^\infty \frac{g(t)dt}{(s^2-t^2)^\alpha}, \quad 0<\alpha<1, \tag{9.64}$$

$$g(t) = -\frac{2\sin\alpha\pi}{\pi}\frac{d}{dt}\int_t^\infty \frac{uf(u)du}{[u^2-t^2]^{1-\alpha}} \tag{9.65}$$

Check your Progress-2

3. Explain **The Generalized Abel's Integral Equation**

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4. State the concept of Inversion formula for singular Integral Equation

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9.6 LET US SUM UP

The Singular integral equation that has enormous applications in applied problems including fluid mechanics, bio-mechanics, and electromagnetic theory.

9.7 KEYWORDS

1. **Monotonic function :** is a **function** which is either entirely nonincreasing or nondecreasing
2. **Hypothesis:** a supposition or proposed explanation made on the basis of limited evidence as a starting point for further investigation.
3. **Notion**: a conception of or belief about something.

9. 8 QUESTIONS FOR REVIEW

1. Solve the following Abel's problem

$$2\sqrt{x} = \int_0^x \frac{1}{\sqrt{x-t}} g(t)dt$$

2. Find an approximate solution to the following Abel's problem

$$\sinh x = \int_0^x \frac{1}{\sqrt{x-t}} g(t)dt$$

In this example ,$f(x)$ = sinh , hence ,$f(0) = 0$ and ,$f'(x) = \cosh x$.

3. Solve

$$f(x) = \int_x^b \frac{y(t)dt}{(\cos x - \cos t)^{\frac{1}{2}}}\ , 0 \le a < x < b \le \pi.$$

9.9 SUGGESTED READINGS AND REFERENCES

1. M. Gelfand and S. V. Fomin. Calculus of Variations, Prentice Hall.
2. Linear Integral Equation: W.V. Lovitt (Dover).
3. Integral Equations, Porter and Stirling, Cambridge.
4. The Use of Integral Transform, I.n. Sneddon, Tata-McGrawHill, 1974
5. R. Churchil& J. Brown Fourier Series and Boundary Value Problems, McGraw-Hill, 1978
6. D. Powers, Boundary Value Problems Academic Press, 1979.

9.10 ANSWERS TO CHECK YOUR PROGRESS

1. Provide definition – 9.2

2. Provide explanation – 9.3

3.Provide explanation – 9.4

4. Provide explanation – 9.5

UNIT 10: VARIATION PRINCIPLE

STRUCTURE

10.0 OBJECTIVES

Understand the concept of Singular integral equation

Comprehend The Euler–Lagrange equation and Hamilton's principle of least action

Understand the Minimal surface of revolution and The brachistochrone Enumerate Geodesics on the sphere

Understand the Isoperimetric Problems

10.1 INTRODUCTION

The calculus of variations gives us precise analytical techniques to answer questions of the following type.

- Find the shortest path (i.e., geodesic) between two given points on a surface.
- Find the curve between two given points in the plane that yields a surface of revolution of minimum area when revolved around a given axis.
-Find the curve along which a bead will slide (under the effect of gravity) in the shortest time.

It also underpins much of modern mathematical physics, via Hamilton's principle of least action. It can be used both to generate interesting differential equations.

10.2 FINDING EXTREMA OF FUNCTIONS OF SEVERAL VARIABLES

We start by introducing some notation. Let $x \in \mathbb{R}^n$ be an arbitrary point. We shall denote by $\mathbb{R}^n_x$ the space of vectors based at the point *x*. The space $\mathbb{R}^n_x$ is called the tangent space to $\mathbb{R}^n$ at the point *x*. Let $U \subset \mathbb{R}^n$ be an open subset and let $f : U \rightarrow$ R be a differentiable function. Recall that a point $x \in U$ is a critical point of the function *f* if $Df(x) = 0$, where $Df(x) \in (\mathbb{R}^n_x)^*$ is the derivative matrix of *f* at *x*.
This condition is equivalent to $D\,f(x)\varepsilon = 0$ for all tangent vectors ε at *x*; that is, for all $\varepsilon \in \mathbb{R}^n_x$ In turn this condition is equivalent to

$$\boxed{\left.\frac{d}{ds} f(x + s\varepsilon)\right|_{s=0} = 0 \qquad \forall \varepsilon \in \mathbb{R}^n_x} \; . \qquad (1)$$

There are three main ingredients in this equation: the point $x \in U \subset \mathbb{R}^n$, a function *f* defined on *U* and the tangent space $\mathbb{R}^n_x$ at *x*. We will now generalise this to functionals.

10.2.1 A motivating example: geodesics

As a motivating example, let us consider the problem of finding the shortest path between two points in the plane: *P* and *Q* say. It is well-known that the answers is the straight line joining these two points, but let us derive this.

By a path between *P* and *Q* we mean a twice continuously differentiable curve (a C^2 curve for short)

$$x : [0, 1] \to \mathbb{R}^2 \qquad t \mapsto (x^1(t), x^2(t))$$

with the condition that $x(0) = P$ and $x(1) = Q$. The arc length of such a path is obtained by integrating the norm of the velocity vector

$$S[x] = \int_0^1 \|\dot{x}(t)\| dt \ ,$$

$$\|\dot{x}(t)\| = \sqrt{(\dot{x}^1(t))^2 + (\dot{x}^2(t))^2} \ .$$

Finding the shortest path between *P* and *Q* means minimising the arclength over the space of all paths between *P* and *Q*. To use equation (1) we need to identify its ingredients in the present problem. The role of $U \subset \mathbb{R}^n$ is played here by the (infinite-dimensional) space of paths in $\mathbb{R}^2$ from *P* to *Q*, and the function to be minimised is the arclength *S*. The final ingredient needed in order to mimic (1) is the analogue of the tangent space $\mathbb{R}^n_x$.

These are the vectors based at *x*, hence they can be understood as differences of points $y - x$ for $y, x \in \mathbb{R}^n$. In our case, they are differences of *C*2 curves $x(t)$ and $y(t)$ from *P* to *Q*. Let $\varepsilon(t) = y(t) - x(t)$ be one such difference of curves. Then $\varepsilon : [0, 1] \to \mathbb{R}^2$ is itself a *C*2 function with the condition that $\varepsilon(0) = \varepsilon(1) = 0 \in \mathbb{R}^2$. Such a ε is called an (endpoint-fixed) variation, hence the name of the theory.

The condition for a path *x* being a critical point of the arclength functional *S* is now given by a formula analogous to (1):

$$\frac{d}{ds} S[x + s\varepsilon]\bigg|_{s=0} = 0$$

As we now show, this condition translates into a differential equation for the path *x*. Notice that

$$\begin{aligned} S[x+s\varepsilon] &= \int_0^1 \|\dot{x}(t)+s\dot{\varepsilon}(t)\| dt \\ &= \int_0^1 \langle \dot{x}(t)+s\dot{\varepsilon}(t), \dot{x}(t)+s\dot{\varepsilon}(t)\rangle^{1/2}\, dt \ , \end{aligned}$$

$$\begin{aligned} \frac{d}{ds}S[x+s\varepsilon] &= \int_0^1 \frac{d}{ds}\langle \dot{x}+s\dot{\varepsilon}, \dot{x}+s\dot{\varepsilon}\rangle^{1/2}\, dt \\ &= \int_0^1 \frac{\langle \dot{x}+s\dot{\varepsilon}, \dot{\varepsilon}\rangle}{\|\dot{x}+s\dot{\varepsilon}\|} dt \ . \end{aligned}$$

Evaluating at $s = 0$, we find

$$\begin{aligned} \frac{d}{ds}S[x+s\varepsilon]\bigg|_{s=0} &= \int_0^1 \frac{\langle \dot{x}, \dot{\varepsilon}\rangle}{\|\dot{x}\|} dt \\ &= \int_0^1 \left\langle \frac{\dot{x}}{\|\dot{x}\|}, \dot{\varepsilon} \right\rangle dt \ . \end{aligned}$$

Integrating by parts and using that $\varepsilon(0) = \varepsilon(1) = 0$, we find that

$$\frac{d}{ds}S[x+s\varepsilon]\bigg|_{s=0} = -\int_0^1 \left\langle \frac{d}{dt}\left(\frac{\dot{x}}{\|\dot{x}\|}\right), \varepsilon \right\rangle dt \ .$$

Therefore a path *x* is a critical point of the arclength functional *S* if and only if

$$\int_0^1 \left\langle \frac{d}{dt}\left(\frac{\dot{x}}{\|\dot{x}\|}\right), \varepsilon \right\rangle dt = 0 \ . \tag{2}$$

We will prove in the next section that this actually implies that

$$\frac{d}{dt}\left(\frac{\dot{x}}{\|\dot{x}\|}\right) = 0 \ , \tag{3}$$

which says that the velocity vector $\dot{x}$ has constant direction; i.e., that it is a straight line. There is only one straight line joining *P* and *Q* and it is clear from the geometry that this path actually minimises arclength.

10.2.2 The fundamental lemma of the calculus of variations

In this section we prove an easy result from analysis which was used above to go from equation (2) to equation (3). This result is fundamental to the calculus of variations.

Theorem 1 (Fundamental Lemma of the Calculus of Variations)**.** *Let f* : [0, 1] → R*n be a continuous function which obeys*

$$\int_0^1 \langle f(t), h(t)\rangle \, dt = 0$$

for all C^2 functions h : [0, 1] → $\mathbb{R}^n$ with h(0) = h(1) = 0. Then f ≡ 0. We will prove the case *n* = 1 and leave the general case as an (easy) exercise.

Proof for n = 1. Let *f* : [0, 1] → R be a continuous function which obeys

$$\int_0^1 f(t)h(t)dt = 0$$

for all C^2 functions *h* : [0, 1] → $\mathbb{R}$ with *h*(0) = *h*(1) = 0. Then we will prove that *f* ≡ 0.Assume for a contradiction that there is a point *t*0 ∈ [0, 1] for which $f(t_0) \neq 0$. We will assume in addition that $f(t_0) > 0$, with a similar proof working in the case $f(t_0) < 0$. Because *f* is continuous, there is a neighbourhood *U* of t_0 in which $f(t) > c > 0$ for all $t \in U$.

We will now construct a *C*2 function *h* : [0, 1] → R with the following properties:

(P1) $h(t) = 0$ for all t outside the neighbourhood U; and

(P2) $\int_0^1 h(t)dt = \int_U h(t)dt > 0$.

Postponing for a moment the construction of such a function, let us see how their existence allows us to prove the Lemma. Let us estimate the integral

$$\begin{aligned} \int_0^1 f(t)h(t)dt &= \int_U f(t)h(t)dt && \text{using (P1)} \\ &> c\int_U h(t)dt && \text{since } f > c \text{ on } U \\ &> 0 && \text{using (P2).} \end{aligned}$$

This violates the hypothesis of the Lemma, hence we deduce that there is no point t_0 for which $f(t_0) \neq 0$.

10.3 THE EULER–LAGRANGE EQUATION

Let $\mathcal{C}_{P,Q}$ be the space of C^2 curves $x : [0, 1] \to \mathbb{R}^n$ with $x(0) = P$ and $x(1) = Q$. Let $L : \mathrm{R}^{2n+1} \to \mathrm{R}$ be a sufficiently differentiable function (typically smooth in applications) and let us consider the functional $S : \mathcal{C}_{P,Q} \to \mathbb{R}$ defined by

$$S[x] = \int_0^1 L(x(t), \dot{x}(t), t)\, dt \ .$$

The function L is called the lagrangian and the functional S is called the action. Extremising S will yield a differential equation for x. Recall that a path x is a critical point for the action if, for all endpoint-fixed variations ε, we have

$$\frac{d}{ds} S[x + s\varepsilon]\Big|_{s=0} = 0 \ .$$

Differentiating under the integral sign, we find

$$
\begin{aligned}
0 &= \int_0^1 \frac{d}{ds} L(x + s\varepsilon, \dot{x} + s\dot{\varepsilon}, t)\Big|_{s=0} dt \\
&= \int_0^1 \left(\sum_{i=1}^{n} \frac{\partial L}{\partial x^i}\varepsilon^i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{x}^i}\dot{\varepsilon}^i \right) dt \\
&= \int_0^1 \sum_{i=1}^{n} \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} \right) \varepsilon^i dt \ ,
\end{aligned}
$$

where we have integrated by parts and used that $\varepsilon(0) = \varepsilon(1) = 0$. Using the Fundamental Lemma, this is equivalent to

$$
\boxed{\frac{\partial L}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}} \ , \qquad (5)
$$

for all $i = 1, 2, \ldots, n$. This is the Euler–Lagrange equation.

As an example, let us reconsider the lagrangian $L(x, \dot{x}, t) = \|\dot{x}\|$Then

$$
\frac{\partial L}{\partial x^i} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \dot{x}^i} = \frac{\dot{x}^i}{\|\dot{x}\|} \ ,
$$

and the Euler–Lagrange equation simply says that $\frac{\dot{x}^i}{\|\dot{x}\|}$, is constant, as we saw above.

Check your Progress-1

1. State and Prove (Fundamental Lemma of the Calculus of Variations)**.**

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2. **Explain** The Euler–Lagrange equation

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10.4 HAMILTON'S PRINCIPLE OF LEAST ACTION

Consider a particle of mass *m* moving in R3 under the influence of a potential $V : \mathbb{R}^3 \to \mathbb{R}$.
Let $x : \mathbb{R} \to \mathbb{R}^3$ denote the trajectory of this particle. Define the kinetic energy of the trajectory to be the function $T : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$T(\dot{x}) = \tfrac{1}{2}m\|\dot{x}\|^2 \ .$$

We define the lagrangian to be the difference between the kinetic and potential energies

$$L(x, \dot{x}) = T(\dot{x}) - V(x) \ .$$

The action of the trajectory from time t_0 to time t_1 is the integral

$$S[x] = \int_{t_0}^{t_1} L(x(t), \dot{x}(t))dt \ .$$

Hamilton's Principle of Least Action says that particles follow trajectories which minimize the action. Such trajectories are therefore called physical trajectories.

For the above Lagrangian, we have

$$\frac{\partial L}{\partial x^i} = -\frac{\partial V}{\partial x^i} \quad \text{and} \quad \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}^i \ ,$$

and the Euler–Lagrange equation is nothing but Newton's second law:

$$m\ddot{x}^i = -\frac{\partial V}{\partial x^i} \ ,$$

where we recognise the right-hand side of this equation as the force due to the potential V .More generally, for any lagrangian (not necessarily of the form $T - V$) one calls the quantity $\partial L/\partial \dot{x}^i$ the force, the quantity $\partial L/\partial \dot{x}^i$ the momentum, and the quantity $\sum_{i=1}^{n} \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} 1 - L$ the energy. For the above Lagrangian $L = T - V$, the energy is $T + V$.

10.4.1 Minimal surface of revolution.

Consider two points in the plane with coordinates (x_1, y_1) and (x_2, y_2) with $x_2 > x_1$. Let $f : [x_1, x_2] \to \mathbb{R}$ be a C^2 function with the property that $f(x_1) = y_1$ and $f(x_2) = y_2$. The graph of this function is a curve from (x_1, y_1) to (x_2, y_2). Now consider revolving this curve around the x-axis to yield a surface of revolution. The surface area of the resulting surface of revolution is given by the following integral

$$S[f] = 2\pi \int_{x_0}^{x_1} f(x)\sqrt{1 + f'(x)^2}\, dx \ ,$$

where $f'(x)$ is the derivative of $f(x)$ with respect to x.

<u>10.4.2 The brachistochrone.</u> Consider a bead of mass m which can slide down a wire frame under the influence of gravity but without any friction. Suppose that the bead is dropped from rest from a height h. Let τ denote the time it takes to slide down to the ground. This time will depend on the shape of the wire. The shape for which τ is minimal is called the *brachistochrone* (Greek for “shortest time”).

We will assume that the wire has no torsion, so that the motion of the bead happens in one plane: the (x, z) plane with z the vertical displacement and x the horizontal displacement. We choose our axes in such a way that wire touches the ground at the origin of the plane: (0, 0). The shape of the wire is given by a function $z = z(x)$, with $z(0) = 0$ and $z(h) = l$. Let s denote the length *along* the wire from the origin to the point (x, z) on the wire.

The kinetic energy of the bead at any time t after being dropped is given by

$$T = \tfrac{1}{2} m \left(\frac{ds}{dt}\right)^2$$

whereas the potential energy is given by

$$V = -mg\,(h - z)$$

Energy is conserved because there is no friction, whence $T + V$ is a constant. To compute it, we evaluate it at the moment the bead is dropped. Because it is dropped from rest, $ds/dt = 0$ and hence $T = 0$. Since the bead is dropped from a height h, the potential energy also vanishes, and we have that $T + V = 0$. From this identity we can solve for ds/dt:

$$\frac{ds}{dt} = -\sqrt{2g\,(h - z)}\ , \qquad (6)$$

where we have chosen the negative sign for the square root, because as the bead falls, s decreases. Now, the length element along the wire is given by

$$ds = dx\sqrt{1 + z'(x)^2}\ . \qquad (7)$$

Let us rewrite equation (6) as

$$dt = -\frac{1}{\sqrt{2g\,(h - z)}}\,ds\ .$$

and insert equation (7) in this equation, to obtain

$$dt = -\frac{\sqrt{1 + z'(x)^2}}{\sqrt{2g\,(h - z(x))}}\,dx\ .$$

Integrating this expression, we obtain the time τ taken by the bead to fall from the point (l, h) to the point $(0, 0)$:

$$\tau = \frac{1}{\sqrt{2g}}\int_0^{\ell}\frac{\sqrt{1 + z'(x)^2}}{\sqrt{(h - z(x))}}\,dx\ .$$

This formula defines a functional on functions $z : [0, l]$, $x \mapsto z(x)$, with $z(0) = 0$ and $z(h) = l$, given by

$$S[z] = \int_0^\ell \frac{\sqrt{1 + z'(x)^2}}{\sqrt{(h - z(x))}}\, dx \ ,$$

where we have conveniently reabsorbed the constant $\sqrt{2g}$ into the functional.

10.5 GEODESICS ON THE SPHERE

Let P and Q be any two distinct points on the unit sphere S^2 in $\mathbb{R}^3$. Let $x : [0, 1] \to S^2 \subset \mathbb{R}^3$ be a C^2 curve from P to Q. In spherical polar coordinates, we can write

$$x(t) = (\cos\theta(t)\sin\phi(t), \sin\theta(t)\sin\phi(t), \cos\phi(t)) \ .$$

The arclength is computed by integrating $||\dot{x}||$. An easy calculation yields

$$\|\dot{x}\| = \sqrt{\dot{\varphi}^2 + (\sin\varphi)^2\dot{\theta}^2} \ ,$$

whence the arclength of the path defines a functional on functions θ and ϕ

$$S[\theta, \varphi] = \int_0^1 \sqrt{\dot{\varphi}^2 + (\sin\varphi)^2\dot{\theta}^2}\, dt \ .$$

The shortest path between P and Q can now be found by extremising the above functional. It is however technically easier to parametrise the path in terms of the angle ϕ itself, in such a way that the path is given by specifying the function $\phi \mapsto \theta(\phi)$. In terms of this function, the arclength functional becomes

$$S[\theta] = \int_{\varphi_P}^{\varphi_Q} \sqrt{1 + (\sin\varphi)^2(\theta')^2}\, d\varphi \ ,$$

where θ' is now the derivative of θ with respect to ϕ.

10.6 ISOPERIMETRIC PROBLEMS

The original isoperimetric problem was posed by the ancient Greeks: find the closed plane curve of a given length that encloses the largest area. They even managed to convince themselves that the intuitive answer (the circle) was correct. The reason this problem is called isoperimetric is that one is maximising the area inside the curve while keeping the perimeter fixed. More generally, an isoperimetric problem is one where one is trying to extremise a functional subject to a (functional) constraint. In this section we will learn how to deal with such constrained extremisation in the context of the variational calculus. Let us start by setting up the classical isoperimetric problem in this context.

Let $x : [0, 1] \to \mathbb{R}^2$ be a C^2 curve which is closed: $x(0) = x(1)$. The area enclosed by the curve is given by the following functional

$$S[x] = \tfrac{1}{2} \int_0^1 (x^1 \dot{x}^2 - x^2 \dot{x}^1) dt \ ,$$

whereas the perimeter of the curve is given by the following functional:

$$A[x] = \int_0^1 \|\dot{x}\| dt \ .$$

The isoperimetric problem is the following: extremise $S[x]$ subject to $A[x] = l$.

Surely you recognise the finite-dimensional analogue to this problem. Let $f, g : U \subset \mathbb{R}^n \to \mathbb{R}$ be functions of n variables. One can then extremise f subject to $g = 0$. As in SVC, one can use the method of Lagrange multipliers. We define a new function $F : U \times R \to R$ of $n+1$ variables (the new variable, typically denoted λ, is the Lagrange multiplier) by $F(x, \lambda) = f(x) - \lambda g(x)$ and one simply extremises F without any constraints. The resulting equations are

$$\frac{\partial F}{\partial \lambda} = 0 \implies g(x) = 0 \qquad \text{and} \qquad \frac{\partial F}{\partial x^i} = 0 \implies \frac{\partial f}{\partial x^i} = \lambda \frac{\partial g}{\partial x^i} \ .$$

The method of Lagrange multipliers extends to the calculus of variations. Suppose that we want to extremise the action

$$S[x] = \int_0^1 L(x, \dot{x}, t)dt$$

on functions $x : [0, 1] \to \mathbb{R}^n$, subject to the constraint

$$A[x] = \int_0^1 K(x, \dot{x}, t)dt = 0 \ .$$

NOTE:

Without loss of generality we have taken the constraint to be $A[x] = 0$ as opposed to $A[x] = c$ for some constant c. Clearly if $A[x] = c$, $A0[x] = A[x] - c = 0$.

The method of Lagrange multipliers says that we should construct a new functional depending in addition on one extra parameter λ (*not* a function, but a constant)

$$\bar{S}[x, \lambda] = S[x] - \lambda A[x] \ ,$$

and extremise $\bar{S}[x, \lambda]$ in the space of functions $x : [0, 1] \to \mathbb{R}^n$. Any solution of the resulting Euler–Lagrange equation will depend on $2n$ constants of integration *and* the parameter λ. These are then fixed by the $2n$ boundary conditions for $x(0)$ and $x(1)$ *and* the constraint $A[x] = 0$.

The only reason we have $2n$ constants of integration is because the lagrangian is first-order; that is, it depends only on x and $\dot{x}$. This means that the resulting Euler–Lagrange equation is a second-order ordinary differential equation for the n component functions of x and hence there are $2n$ constants of integration: 2 constants per component function. In general, if the lagrangian depends on x and its first k derivatives, we will have kn constants of integration and an equal number of boundary conditions.

Recall that a function $x : [0, 1] \to \mathbb{R}^n$ is a critical point of the functional $S[x]$ if for any variation ε.

$$\frac{d}{ds}S[x+s\varepsilon]\bigg|_{s=0}=0\ .$$

In the presence of a constraint $A[x] = 0$, we would have to consider only those variations which preserve the constraint; that is, only those ε for which $A[x + s\varepsilon] = 0$ for all s. This condition is generally too strong and there may not be any nontrivial variations satisfying this. Instead we introduce a two-parameter family of variations: $S[x + s\varepsilon + r\eta]$ and we choose the parameters s and r in such a way that $A[x + s\varepsilon + r\eta] = 0$. At a fixed function x and for fixed variations $\varepsilon$and η, the condition $A[x + s\varepsilon + r\eta] = 0$ defines a curve in the (r, s) plane: $g(r, s) = 0$. Hence, for fixed x, ε, η, we want to extremise the function $f(r, s) = S[x + s\varepsilon + r\eta]$ subject to the condition $g(r, s) = 0$.

The method of Lagrange multipliers for functions of two variables (here s and r) says that we should extremise the function

$$F(r,s,\lambda)=f(r,s)-\lambda g(r,s)\ ,$$

which is nothing but

$$F(r,s,\lambda)=S[x+s\varepsilon+r\eta]-\lambda A[x+s\varepsilon+r\eta]\ .$$

This function has a critical point if the following conditions are satisfied:

$$\frac{\partial F}{\partial r}\bigg|_{r=s=0}=\frac{\partial F}{\partial s}\bigg|_{r=s=0}=0\qquad\text{and}\qquad\frac{\partial F}{\partial\lambda}\bigg|_{r=s=0}=0\ .$$

Check your Progress-2

3. Discuss Hamilton's principle of least action

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4. What do you understand by Geodesics on the sphere?

10.7 LET US SUM UP

Thus far we have considered functionals defined on curves; that is, on functions of one variable. The Euler–Lagrange equations obtained in this way are always ordinary differential equations. In the same way, one can obtain partial differential equations by varying functional of functions of several variables. In fact, many of the interesting partial differential equations arise in this way.

10.8 KEYWORDS

1. Extremise Function: In calculus of variations the basic problem is to find a **function** y for which the **functional** I(y) is maximum or minimum. We call such **functions** as **extremizing functions** and the value of the **functional** at the **extremizing function** as extremum
2. A **second order ordinary differential equation** is an **ordinary differential equation** in which any derivatives with respect to the independent variable have **order** no greater than 2.

10.9 QUESTIONS FOR REVIEW

1. *Prove that θ so defined is a smooth function.*

$$\theta(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0\,. \end{cases}$$

2. Extremising the functional S[θ], prove that the shortest path between any two points P and Q on the unit sphere lies on a great circle; that is, on the intersection of the sphere with a plane through the centre of the sphere

Where

$$S[\theta] = \int_{\varphi_P}^{\varphi_Q} \sqrt{1 + (\sin\varphi)^2(\theta')^2}\, d\varphi \ ,$$

10.10 SUGGESTED READINGS AND REFERENCES

1. M. Gelfand and S. V. Fomin. Calculus of Variations, Prentice Hall.
2. Linear Integral Equation: W.V. Lovitt (Dover).
3. Integral Equations, Porter and Stirling, Cambridge.
4. The Use of Integral Transform, I.n. Sneddon, Tata-McGrawHill, 1974
5. R. Churchil& J. Brown Fourier Series and Boundary Value Problems, McGraw-Hill, 1978
6. D. Powers, Boundary Value Problems Academic Press, 1979.

10.11 ANSWERS TO CHECK YOUR PROGRESS

1. Provide statement and proof – 10.2.2

2. Provide explanation – 10.3

3. Provide explanation – 10.4

4. Provide explanation – 10.5

UNIT 11: INTEGRAL TRANSFORM METHOD I

STRUCTURE

11.0 OBJECTIVES

Understand the concept of Integral Transform

Comprehend Laplace Transform

Enumerate Convolution Theorem

Understand the Applications to Volterra Integral Equations with Convolution-Type Kernels

11.1 INTRODUCTION

Integral transforms are used in handling of partial differential equations. The choice of a particular transform to be employed for the solution of an equation depends on the boundary conditions of the problem and the ease with which the inverse transform can be obtained. An integral transform when applied to a partial differential equation (PDE), reduces its number of independent variable by one. They are generally applied to the problems related to the transmission lines, conduction of heat, vibrations of a string, transverse oscillations of an elastic beam, free and forced oscillations of a membrane, etc.

11.2 INTEGRAL TRANSFORM

The integral transform of $f(x)$ defined by *I[f(x)]* is given by *f(s)* = $\int_a^b K(s,x)f(x)dx$ where K *(s, x)* is called the kernel of the transformation, a known function of *s* and *x*. The limits a and b may be finite or infinite, and when at least one limit is infinite, this integral becomes improper. The function is called the inverse transform of *f (s)*.

Depending upon the type of kernel and the limits, we can obtain various types of integral transforms, e.g. Laplace transform, Fourier transform, Mellin transform, Hankel transform, Legendre transform, Laguerre transform, etc. There are some examples of the kernel as follows:

(i) When $K(s,x) = e^{-sx}$, the Laplace transform of $f(x)$ is $L(s) = \int_0^\infty f(x)e^{-sx}dx$.

(ii) When $K(s,x) = e^{isx}$, the Fourier transform of $f(x)$ is $F(s) = \int_{-\infty}^\infty f(x)e^{isx}dx$.

(iii) When $K(s,x) = x^{s-1}$, the Fourier transform of $f(x)$ is $M(s) = \int_0^\infty f(x)x^{s-1}dx$.

When kernel is sine or cosine or Bessel's function, the transformation is called Fourier sine or Fourier cosine or Hankel transform, respectively. The integral transform methods are of great value in the treatment of integral equations, especially the singular integral equations. Suppose that a relationship of the form

$$g(s) = \int \int \Gamma(s,x)K(x,t)g(t)dt\,dx \qquad (11.1)$$

is known to be valid and that this double integral can be evaluated as an iterated integral. This means that the solution of the integral equation of the first kind,

$$f(s) = \int K(x,t)g(t)dt \qquad (11.2)$$

$$g(s) = \int \Gamma(s,x)f(t)dt \qquad (11.3)$$

Conversely, the relation (11.2) can be considered as the solution of the integral equation (11.3). It is conventional to refer to one of these functions as the transform of the second and to the second as an inverse transform of the first. The most celebrated example of the double integral (11.1) is the Fourier integral

$$g(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{isx}e^{-ixt}g(t)dt\,dx \qquad (11.4)$$

which results in the reciprocal relations

$$f(s) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{-ist}g(t)dt \qquad (11.5)$$

and

$$g(s) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{ist}f(t)dt \qquad (11.6)$$

The function *f(s)* is known as the Fourier transform) T[g] of *g(t)* and *g(s)* as the inverse transform T^{-1} [*f*]of *f(s),* and vice versa. The function *f(s)* exists if *g(t)* is absolutely integrable, and it is square-integrable if *g(t)* is square-integrable, as can be readily verified using Bessel's inequality. In the sequel, we shall assume that the functions involved in the integral equations as well as their transforms satisfy the appropriate regularity conditions, so that the required operations are valid.

As a second example, consider the double integral

$$g(s) = \left(\frac{2}{\pi}\right) \int_0^\infty \int_0^\infty \sin sx \sin xt \, g(t) dt dx \qquad (11.7)$$

This leads to the sine transform and its inverse,

$$f(s) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \sin st \, g(t) dt \qquad (11.8)$$

$$g(s) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \sin sx \, f(t) dt \qquad (11.9)$$

respectively. For ease of notation, we shall also call the transform of *f* as F and that of *g* as G, etc., for all the transforms. It will be clear in the context as to what transforms we are implying.

11.3 LAPLACE TRANSFORM

Laplace transform is a powerful tool for solving linear differential equations. Laplace transform converts a linear differential equation to an algebraic problem. This process of changing from operations of calculus to algebraic operations on transforms in solving initial value problem is known as operational calculus, which is an important area of applied mathematics.

The advantage of Laplace transforms in solving initial value problems lies in the fact that the initial conditions are taken care of at the outset and the solution is directly obtained without resorting to finding the general solution and then the arbitrary constants. The name is due to the French mathematician Pierre Simon de Laplace who used this transform while developing the theory of probability.

The Laplace transform L[*f*] of a function *f(s)* is defined as

$$L[f] = F(p) = \int_0^\infty f(s)e^{-ps}ds \qquad (11.10)$$

And for *L[f]* = F(p) then *f* is called an Inverse Laplace Transform of F(*p*), and we write

$$L^{-1}[F] = f(s) \qquad (11.11)$$

L^{-1} is known as the inverse Laplace Transformation operator.

11.3.1 Some Useful Results about Laplace Transform

Table of Laplace transform of some elementary functions :

3.	t^n(n is a positive integer)	$\frac{n!}{p^{n+1}}, p > 0$
4.	e^{at}	$\frac{1}{p-a}, \quad p > a$
5.	$\sin at$	$\frac{a}{p^2+a^2}, \quad p > 0.$
6.	$\cos at$	$\frac{p}{p^2+a^2}, \quad p > 0.$
7.	$\sinh at$	$\frac{a}{p^2-a^2}, \quad p > \|a\|.$
8.	$\cosh at$	$\frac{p}{p^2-a^2}, \quad p > \|a\|.$
9.	$\delta(t-a)$	e^{-ap}

Notes

1. **Linearity property of Laplace transforms.** If $c_1$and c_2 be constants, then

$$L\{c_1F_1(t) + c_2F_2(t)\} = c_1L\{F_1(t)\} + c_2L\{F_2(t)\}$$

3. First translation (or shifting) theorem.

If $L\{F(t)\} = F(s)$then $L\{e^{at}f(t)\} = F(s-a)$

4. **Unit step function or Heaviside's unit function.**

Definition. It is denoted and defined as

$$H(t-a) = \begin{cases} 0, & \text{if } t < a \\ 1, & \text{if } t \geq a. \end{cases}$$

Note :$L\{H(t-a)\} = \frac{1}{p} \times e^{-ap}$

5. Second translation (or shifting) theorem

$$If\ L\{F(t)\} = \overline{F}(p)\ then\ L\{F(t-a)H(t-a)\} = e^{-ap}\overline{F}(p)$$

6. Change of scale property

$$If\ L\{F(t)\} = \overline{F}(p)\ then\ L\{F(at)\} = \left(\frac{1}{a}\right) \times \overline{F}\left(\frac{p}{a}\right)$$

7. **Laplace transform of derivatives:**

(i) $L\{F'(t)\} = pL\{F(t)\} - F(0)$: In particular, if$F(0) = 0$ then$L\{F'(t)\} = pL\{F(t)\}$

(ii) $L\{F''(t)\} = p^2L\{F(t)\} - pF(0) - F'(0)$and so on.

8. **Multiplication by positive integral powers of t.**

(i) If $L\{F(t)\} = \overline{F}(p)$, then $L\{tf(t)\} = -\frac{d}{dp}\overline{F}(p)$

(ii) If $L\{F(t)\} = \overline{F}(p)$, then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{dp^n}\overline{F}(p)$

9. Division by t.

If $L\{F(t)\} = \overline{F}(p)$, then $L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty \overline{F}(p)dp$, provided the integral exists

Initial value theorem : $\lim_{t\to 0} F(t) = \lim_{p\to\infty} p\overline{F}(p).$

Final value theorem : $\lim_{t\to\infty} F(t) = \lim_{p\to 0} p\overline{F}(p).$

10. Laplace transform of periodic function. Given that F(t) is a periodic function with period a , that is, F(t + na) = F(t), for n = 1, 2, 3, . .. Then, we have

$$L\{F(t)\} = \frac{1}{1-e^{-ap}}\int_0^a e^{-ap}F(t)dt$$

11. Table of inverse Laplace transform of some functions

S.No.	$\overline{F}(p)$	$L^{-1}\{\overline{F}(p)\}$
1.	$\frac{1}{p}$	1
2.	$\frac{1}{p^{n+1}}$, $n > -1$	$\frac{t^n}{\Gamma(n+1)}$
3.	$\frac{1}{p^{n+1}}$ (n is a positive integer)	$\frac{t^n}{n!}$
4.	$\frac{1}{p-a}$	e^{at}
5.	$\frac{1}{p^2+a^2}$	$\frac{\sin at}{a}$
6.	$\frac{p}{p^2+a^2}$	$\cos at$
7.	$\frac{1}{p^2-a^2}$	$\frac{\sinh at}{a}$
8.	$\frac{p}{p^2-a^2}$	$\cosh at$

Notes

12. Laplace Transform of Integral If f(t) is a piecewise continuous in every finite interval $0 \le t \le a$ in $[0, \infty)$ and f(t) is of exponential order $\alpha > 0$ and if L[f(t)] = F(s), then

$$L\left[\int_0^t f(u)du\right] = \frac{F(s)}{s} \text{for } s > \alpha$$

Check your Progress-1

1. What is Integral Transform?

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2.Define **Unit step function**

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11.4 CONVOLUTION THEOREM

11.4.1 Method of Convolution

Let f(t), g(t) be two functions defined for all $t \ge 0$. The convolution of f(t) and g(t) is defined as the integral

$$f(t) * g(t) = \int_0^t f(u)g(t-u)du$$

Note: $f(t) * g(t) = g(t) * f(t)$

11.4.2 Integral equation of convolution type T

he integral equation

$$y(t) = f(t) + \int_0^t K(t-x)f(x)dx \qquad (11.12)$$

in which the kernel K($t-x$) is a function of the difference ($t-x$) only, is known as integral equation of the convolution type. Using the definition of convolution, we may re-write it as

$$y(t) = f(t) + K(t) * y(t).$$

11.4.3 Convolution Theorem:

If $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$

Then $L[F(s)G(s)] = L[f(t)]L[g(t)] = F(s)G(s)$

Proof: we have $f(t) * g(t) = \int_0^t f(u)g(t-u)du$

$$\therefore \qquad L[f(t) * g(t)] = \int_0^\infty e^{-st}[f(t) * g(t)]dt$$

$$= \int_0^\infty e^{-st}\left[\int_0^t f(u)g(t-u)du\right]dt$$

$$\Rightarrow \qquad L[f(t) * g(t)] = \int_0^\infty \int_0^t e^{-st} f(u)g(t-u)du\,dt$$

The region of this double integral is bounded by the lines u = 0, u = t, t = 0, and t = ∞.

Changing the order of the integration t varies from u to ∞ and u varies from 0 to ∞.

$$L[f(t) * g(t)] = \int_0^\infty f(u)\left[\int_u^\infty e^{-st} g(t-u)dt\right]du$$

Put v = t − u in the inner integral ∴ $dv = dt$.

When t = u, v = 0 and when t = ∞, v = ∞

$$\int_u^\infty e^{-st} g(t-u)dt = \int_0^\infty e^{-s(v+u)} g(v)dv$$

$$= e^{-su}\int_0^\infty e^{-sv} g(v)dv$$

$$L[f(t) * g(t)] = \int_0^\infty f(u)\left\{e^{-su}\int_0^\infty e^{-sv} g(v)dv\right\}du$$

$$= \int_0^\infty e^{-su} f(u)du \int_0^\infty e^{-sv} g(v)dv$$

$$= L[f(t)]L[g(t)] = F(s)G(s)$$

Or equivalently

$$L^{-1}[F(s)G(s)] = f(t) * g(t) = L^{-1}[F(s)] * L^{-1}[G(s)]$$

Examples 1. Find using convolution theorem

$$L^{-1}\left[\frac{1}{s(s^2-a^2)}\right]$$

We can write $\frac{1}{s(s^2-a^2)} = \frac{1}{s}.\frac{1}{(s^2-a^2)}$

$\therefore \qquad F(s) = \frac{1}{s}$ and $G(s) = \frac{1}{(s^2-a^2)}$

$$L^{-1}\left[\frac{1}{s(s^2-a^2)}\right] = L^{-1}\left[\frac{1}{s}\right] * L^{-1}\left[\frac{1}{(s^2-a^2)}\right]$$

$$= 1 * \frac{1}{a}\sinh at = \frac{1}{a}\int_0^t \sinh au \,.\, 1\, du$$

Here $(t) = \sinh at$, $g(t) = 1 \therefore \quad f(u)g(t-u) = f(u).1 = \sinh au$

$$\therefore L^{-1}\left[\frac{1}{s(s^2-a^2)}\right] = \frac{1}{a}\left[\frac{\cosh au}{a}\right]_0^t = \frac{1}{a^2}[\cosh at - \cosh 0] =$$

$$= \frac{1}{a^2}[\cosh at - 1]$$

11.5 APPLICATION TO VOLTERRA WITH CONVOLUTION – TYPE KERNELS

Consider the Volterra-type integral equation of the first kind,

$$f(s) = \int_0^s k(s-t)g(t)dt, \qquad (11.13)$$

Where k(s − t) depends only on the difference (s − t). Applying the Laplace transform to both sides of this equation, we obtain

$$F(p) = K(p)G(p)$$

$$G(p) = \frac{F(p)}{K(p)} \qquad (11.14)$$

The solution follows by inversion. The present method is also applicable to the Volterra integral equation of the second kind with a convolution-type kernel

$$g(s) = f(s) + \int_0^s k(s-t)g(t)dt, \qquad (11.15)$$

On applying Laplace transformation to both sides and using the convolution formula, we have and inversion yields the solution.

$$G(p) = F(p) + K(p)G(p)$$

$$G(p) = F(p)/[1 - K(p)] \qquad (11.16)$$

We can also find the resolvent kernel of the integral equation (11.15) by integral transform methods. For this purpose, we first show that, if the original kernel k(s, t) is a difference kernel, then so is the resolvent kernel. Since the resolvent kernel Γ (s, t) is a sum of the iterated kernels, all that we have to prove is that they all depend on the difference (s −t). Indeed,

$$k_2(s,t) = \int_t^s k(s-x)k(x-t)dx = \int_0^{s-t} k(s-t-\sigma)k(\sigma)d\sigma \qquad (11.17)$$

where we have set a = x - t. This process can obviously be continued and our assertion is proved. Hence, the solution of the integral equation (11.15) is

$$g(s) = f(s) + \int_0^s \Gamma(s-t)f(t)dt \qquad (11.18)$$

Application of the Laplace transform to both sides of (11.18) gives

$$G(p) = F(p) + \Omega(p)F(p) \qquad (11.19)$$

$$\Omega(p) = L[\Gamma(s-t)] \qquad (11.20)$$

From (11.16) and (11.19), we have

$$\frac{F(p)}{[1-K(p)]} = F(p)[1+\Omega(p)] \qquad (11.21)$$

$$\Omega(p) = \frac{K(p)}{[1-K(p)]} \qquad (11.22)$$

By inversion, we recover Γ (s, t) .

Example 1. Solve the Abel integral equation

$$f(s) = \int_0^s \left[\frac{g(t)}{(s-t)^\alpha}\right] dt \qquad (11.23)$$

This is a convolution integral and therefore

$$F(p) = K(p)G(p) \qquad (11.24)$$

Where K(p) is the Laplace transform of $k(s) = s^{-\alpha}$

$$K(p) = p^{\alpha-1}\Gamma(1-\alpha) \qquad (11.25)$$

From (11.24) and (11.25), it follows that

$$G(p) = \frac{p^{1-\alpha}F(p)}{\Gamma(1-\alpha)} = \frac{p}{\Gamma(\alpha)\Gamma(1-\alpha)}\{\Gamma(\alpha)p^{-\alpha}F(p)\}$$

$$= \frac{p}{\pi \csc \pi\alpha}\{\Gamma(\alpha)p^{-\alpha}F(p)\}$$

where we have used the relation $\Gamma(\alpha)\Gamma(1-\alpha) = \pi \csc \pi\alpha$. Now if we use the relation (11.14), (11.25) becomes

$$G(p) = \frac{\sin \alpha\pi}{\pi} pL\left[\int_0^s (s-t)^{\alpha-1} f(t)dt\right] \qquad (11.26)$$

By virtue of the property 7 of Laplace Transform, we finally have

Example 2. Solve the integral equation

$$s = \int_0^s e^{s-t} g(t)dt \qquad (11.28)$$

$$g(s) = \frac{\sin \alpha\pi}{\pi}\frac{d}{ds}\int_0^s (s-t)^{\alpha-1} f(t)dt \qquad (11.27)$$

Taking the Laplace transform of both sides, we obtain

$$\frac{1}{p^2} = K(p)G(p) \qquad (11.29)$$

Notes

Where K(p) is the Laplace transform of $k(s) = e^s$

$$K(p) = \int_0^\infty e^s e^{-sp} ds = \frac{1}{p-1} \qquad (11.30)$$

The result of combining (11.28), (11.29), (11.30) is

$$G(p) = \frac{p-1}{p^2} = \frac{1}{p} - \frac{1}{p^2}$$

Whose inverse is

$$g(s) = 1 - s$$

Check your Progress-2

1. What do you understand by Method of Convolution**?**

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2.State and prove Convolution Theorem

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11.6 LET US SUM UP

The advantage of Laplace transforms in solving initial value problems lies in the fact that the initial conditions are taken care of at the outset and the solution is directly obtained without resorting to finding the general solution and then the arbitrary constants.

11.7 KEYWORDS

Arbitrary constant. mathematics. : a symbol to which various values may be assigned but which remains unaffected by the changes in the values of the variables of the equation

Periodic function: a function returning to the same value at regular intervals

11.8 QUESTIONS FOR REVIEW

1. With the help of the Laplace transform, solve the below equation for a general convolution kernel

$$f(s) = \int_0^s k(s^2 - t^2)g(t)dt\ , \qquad s > 0\ ,$$

2. Solve the inhomogeneous integral equation

$$g(s) = 1 - \int_0^s (s - t)g(t)dt$$

$$k(s) = s, \quad K(p) = \frac{1}{p^2}\ ,$$

3. Find the resolvent of the integral equation

$$g(s) = f(s) + \int_0^s (s - t)g(t)dt$$

11.9 SUGGESTED READINGS AND REFERENCES

1. M. Gelfand and S. V. Fomin. Calculus of Variations, Prentice Hall.
2. Linear Integral Equation: W.V. Lovitt (Dover).
3. Integral Equations, Porter and Stirling, Cambridge.

4. The Use of Integral Transform, I.n. Sneddon, Tata-McGrawHill, 1974
5. R. Churchil& J. Brown Fourier Series and Boundary Value Problems, McGraw-Hill, 1978
6. D. Powers, Boundary Value Problems Academic Press, 1979.

11.10 ANSWERS TO CHECK YOUR PROGRESS

1. Provide definition and explanation– 11.2

2. Provide definition – 11.3.1

3. Provide definition and expression – 11.4.1

4. Provide statement and proof – 11.4.3

UNIT 12: INTEGRAL TRANSFORM METHOD II

STRUCTURE

12.0 OBJECTIVES

Understand the concept of Fourier Transform and its different properties

Comprehend Mellin transformation

12.1 INTRODUCTION

The **Mellin transform** is an integral transform that may be regarded as the multiplicative version of the two-sided Laplace transform. This integral transform is closely connected to the theory of Dirichlet series, and is often used in number theory, mathematical statistics, and the theory of asymptotic expansions; it is closely related to the Laplace transform and the Fourier transform, and the theory of the gamma function and allied special functions.

12.2 FOURIER TRANSFORM

$$\text{Let } L\{Y\} = \overline{Y}(p), \quad L\{K\} = \overline{K}(p) \text{ and } L\{F\} = \overline{F}(p)$$

Definition:

Given a function Y(x) defined for all x in the interval $-\infty < x < \infty$, the Fourier transform of Y(x) is a function of a new variable 0 given by

$$F\{Y(x)\} = \overline{Y}(p) = \int_{-\infty}^{\infty} e^{ipx} Y(x) dx \qquad (12.1)$$

The function Y(x) is then called inverse Fourier transform of F {Y(x) }or $\bar{Y}(p)$ and is written as Y(x) = F – 1{F(Y(x))}, and is given by

$$Y(x) = F^{-1}\{\overline{Y}(p)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipx} \overline{Y}(p) dp \qquad (12.2)$$

Remark 1. Some authors also define (12.1) and (12.2) in the following manner:

$$F\{Y(x)\} = \overline{Y}(p) = \int_{-\infty}^{\infty} e^{-ipx} Y(x) dx \qquad (12.3)$$

$$Y(x) = F^{-1}\{\overline{Y}(p)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} \overline{Y}(p) dp \qquad (12.4)$$

Remark 2. Some authors also define (12.1) and (12.2) in the so called symmetric form as follows.

$$F\{Y(x)\} = \overline{Y}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} Y(x) dx \qquad (12.5)$$

$$Y(x) = F^{-1}\{\overline{Y}(p)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} \overline{Y}(p) dp \qquad (12.6)$$

12.2.1 The Fourier Sine Transform

The Fourier sine transform of Y(x), $0 < x < \infty$ is denoted and defined as follows :

$$F_s\{Y(x)\} = \overline{Y}_s(p) = \int_0^{\infty} Y(x) \sin px \, dx \qquad (12.7)$$

Then, the corresponding inversion formula is given by

$$Y(x) = F_s^{-1}\{\overline{Y}_s(p)\} = \frac{2}{\pi} \int_0^{\infty} \overline{Y}_s(p) \sin px \, dp \qquad (12.8)$$

12.2.2 The Fourier Cosine Transform:

The Fourier cosine transform of Y(x), $0 < x < \infty$ is denoted and defined as follows :

$$F_c\{Y(x)\} = \overline{Y}_c(p) = \int_0^\infty Y(x)\cos px\,dx \tag{12.9}$$

Then, the corresponding inversion formula is given by

$$Y(x) = F_c^{-1}\{\overline{Y}_s(p)\} = \frac{2}{\pi}\int_0^\infty \overline{Y}_c(p)\cos px\,dp \tag{12.10}$$

12.2.3 Linearity Property of Fourier Transforms:

(i) $F\{c_1Y_1(x) + c_2Y_2(x)\} = c_1F\{Y_1(x)\} + c_2F\{Y_2(x)\}$

(ii) $F_s\{c_1Y_1(x) + c_2Y_2(x)\} = c_1F_s\{Y_1(x)\} + c_2F_s\{Y_2(x)\}$

(iii) $F_c\{c_1Y_1(x) + c_2Y_2(x)\} = c_1F_c\{Y_1(x)\} + c_2F_c\{Y_2(x)\}$

12.2.4 Change of scale property.

(i) If $F\{Y(x)\} = \overline{Y}(p)$, then $F\{Y(ax)\} = \frac{1}{a}\overline{Y}\left(\frac{p}{a}\right)$

(ii) If $F_s\{Y(x)\} = \overline{Y_s}(p)$, then $F_s\{Y(ax)\} = \frac{1}{a}\overline{Y}_s\left(\frac{p}{a}\right)$

(iii) If $F_c\{Y(x)\} = \overline{Y}_c(p)$, then $F_c\{Y(ax)\} = \frac{1}{a}\overline{Y}_c\left(\frac{p}{a}\right)$

12.2.5 Convolution.

The convolution of two functions G(x) and H(x), where $-\infty < x < \infty$, is denoted and defined as

$$G * H = \int_{-\infty}^{\infty} G(x)H(t-x)dx \text{ or } G * H = \int_{-\infty}^{\infty} G(t-x)H(x)dx$$

12.2.6 Convolution property

The Fourier transform of the convolution of G(x) and H(x) is the product of two transforms of G(x) and H(x) i.e.

$$F\{G * H\} = F\{G\}F\{H\}$$

12.2.7 Shifting Property

If $\bar{Y}$(p) is the complex Fourier transform of Y(x), then complex Fourier transform of

$$Y(x-a) \text{ is } e^{ipa}\bar{Y}(p).$$

12.3 FOURIER APPLICATION OF FOURIER TRANSFORM

The whole procedure will be clear from the following examples

Example 1. Solve the integral equation

$$\int_0^\infty F(x)\cos px\,dx = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$$

Let

$$F_c(p) = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases} \qquad (12.11)$$

Then the given integral equation can be re-written as

$$F_c(p) = \int_0^\infty F(x)\cos px\,dx \qquad (12.12)$$

By definition of fourier cosine transform, we see that F_c (p) is the Fourier cosine transform of F(x). Hence, using the corresponding inversion formula, we have

$$F(x) = \frac{2}{\pi}\int_0^\infty F_c(p)\cos px\,dp = \frac{2}{\pi}\left[\int_0^1 F_c(p)\cos px\,dp + \int_1^\infty F_c(p)\cos px\,dp\right]$$

$$= \frac{2}{\pi}\left[\int_0^1 (1-p)\cos px\,dp + \int_1^\infty (0)\times\cos px\,dp\right] \text{using (6.66)}$$

$$= \frac{2}{\pi}\left\{\left[(1-p)\frac{\sin px}{x}\right]_0^1 - \int_0^1 (-1)\times\frac{\sin px}{x}dp\right\} = \frac{2}{\pi x}\int_0^1 \sin px\,dp$$

$$= \frac{2}{\pi x}\left[-\frac{\cos px}{x}\right]_0^1 = \frac{2}{\pi x^2}(-\cos x + 1) = \frac{2(1-\cos x)}{\pi x^2}$$

Example 2. Solve the integral equation

$$\int_0^\infty F(x)\sin xp\,dx = \begin{cases} 1, & 0 \le p \le 1 \\ 2, & 1 \le p < 2 \\ 0, & p \ge 2 \end{cases}$$

$$\bar{F}_s\ (p) = \begin{cases} 1, & 0 \le p \le 1 \\ 2, & 1 \le p < 2 \\ 0, & p \ge 2 \end{cases} \qquad (12.13)$$

Then the given integral equation can be re-written as

$$\bar{F}_s\ (p) = \int_0^\infty F(x)\sin px\,dx \qquad (12.14)$$

By definition of Fourier sine transform, we see that $\bar{F}$s(p) is the Fourier cosine transform of F(*x*). Hence, using the corresponding inversion formula, we have

$$F(x) = \frac{2}{\pi}\int_0^\infty \bar{F}_s\ (p)\sin px\,dp$$

$$= \frac{2}{\pi}\left[\int_0^1 \bar{F}_s\ (p)\sin px\,dp + \int_1^2 F(x)\sin px\,dx + \int_2^\infty F(x)\sin px\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^1 \sin px\,dp + \int_1^2 2\times\sin px\,dx + \int_2^\infty 0\times\sin px\,dx\right]$$

$$\frac{2}{\pi}\left\{\left[-\frac{\cos px}{x}\right]_0^1 + 2\left[-\frac{\cos px}{x}\right]_1^2\right\} = \frac{2}{\pi}\left\{\frac{-\cos x + 1}{x} + 2\frac{-\cos 2x + \cos x}{x}\right\}$$

$$\left(\frac{2}{\pi x}\right)\times(1 + \cos x - 2\cos 2x).$$

Check your Progress-1

1. Explain Fourier Transform

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2.Define Convolution

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12.3 MELLIN TRANSFORMATION

Let *f*(t) be a function defined on the positive real axis 0 < t < ∞. The Mellin transformation $\mathcal{M}$ is the operation mapping the function *f* into the function F defined on the complex plane by the relation:

$$\mathcal{M}[f;s] \equiv F(s) = \int_0^{\infty} f(t)\, t^{s-1}\, dt \qquad (12.15)$$

The function *F*(*s*) is called the Mellin transform of *f*. In general, the integral does exist only for complex values of *s* = *a* + *jb* such that *a* < *a*1 < *a*2, where *a*1 and *a*2 depend on the function *f(t)* to transform. This introduces what is called the *strip of definition* of the Mellin transform that will be denoted by *S*(*a*1, *a*2).

In some cases, this strip may extend to a half-plane (*a*1 = –∞ or *a*2 = + ∞) or to the whole complex *s*-plane (*a*1 = –∞ and *a*2 = + ∞)

Example: Consider:

$$f(t) = H(t - t_0)\, t^z$$

where *H* is Heaviside's step function, *t*0 is a positive number and *z* is complex. The Mellin transform of *f* is given by:

$$\mathcal{M}\left[f;\,s\right]=\int_{t_0}^{\infty} t^{z+s-1}\,dt=-\frac{t_0^{z+s}}{z+s}$$

provided s is such that $Re(s) < -Re(z)$. In this case the function $f(s)$ is holomorphic in a half-plane.

12.3.1 Relation to Laplace and Fourier Transformations

Mellin's transformation is closely related to an extended form of Laplace's. The change of variables defined by:

$$t=e^{-x},\quad dt=-e^{-x}\,dx \tag{12.16}$$

$$F(s)=\int_{-\infty}^{\infty} f\left(e^{-x}\right)e^{-sx}\,dx \tag{12.17}$$

Transforms the integral (12.15) into:

After the change of function:

$$g(x)\equiv f\left(e^{-x}\right) \tag{12.18}$$

the *two-sided* Laplace transform of *g* usually defined by:

$$\mathfrak{L}\left[g;\,s\right]=\int_{-\infty}^{\infty} g(x)\,e^{-sx}\,dx \tag{12.19}$$

This can be written symbolically as:

$$\mathcal{M}\left[f(t);\,s\right]=\mathfrak{L}\left[f\left(e^{-x}\right);\,s\right] \tag{12.20}$$

The occurrence of a strip of holomorphic for Mellin's transform can be deduced directly from this relation. The usual right-sided Laplace transform is analytic in a half-plane $Re(s) > \sigma 1$. In the same way, one can define a left-sided Laplace transform analytic in the region $Re(s) < \sigma 2$. If the two half-planes overlap, the region of holomorphy of the two-sided transform is thus the strip $\sigma 1 < Re(s) < \sigma 2$ obtained as their intersection.

To obtain Fourier's transform, write now $s = a + 2\pi j\beta$

$$F(s) = \int_{-\infty}^{\infty} f(e^{-x})\, e^{-ax}\, e^{-j2\pi\beta x}\, dx \qquad (12.21)$$

The result is

$$\mathcal{M}[f(t); a + j2\pi\beta] = \Im[f(e^{-x})\, e^{-ax}; \beta] \qquad (12.22)$$

where $\Im$ represents the Fourier transformation defined by:

$$\Im[f;\beta] = \int_{-\infty}^{\infty} f(x)\, e^{-j2\pi\beta x}\, dx \qquad (12.23)$$

Thus, for a given value of $Re(s) = a$ belonging to the definition strip, the Mellin transform of a function can be expressed as a Fourier transform

12.3.2 Inversion Formula

A direct way to invert Mellin's transformation (12.15) is to start from Fourier's inversion theorem. As is well know, if $\check{f} = \Im[f;\beta]$ is the Fourier transform (12.23) of f, the original function is recovered by:

$$f(x) = \int_{-\infty}^{\infty} \check{f}(\beta)\, e^{j2\pi\beta x}\, d\beta \qquad (12.24)$$

Applying this formula to (12.21) with $s = a + j2\pi\beta$ yields:

$$f(e^{-x})\, e^{-ax} = \int_{-\infty}^{\infty} F(s)\, e^{j2\pi\beta x}\, d\beta \qquad (12.25)$$

Hence, going back to variables t and s:

$$f(t) = t^{-a} \int_{-\infty}^{\infty} F(s)\, t^{-j2\pi\beta}\, d\beta \qquad (12.26)$$

The inversion formula finally reads:

$$f(t) = (1/2\pi j) \int_{a-j\infty}^{a+j\infty} F(s)\, t^{-s}\, ds \qquad (12.27)$$

where the integration is along a vertical line through *Re*(*s*) = *a*. Here a few questions arise. What value of *a* has to be put into the formula? What happens when *a* is changed? Is the inverse unique? In what case is *f* a function defined for all *t*'s?

It is clear that if *F* is holomorphic in the strip *S*(*a*1, *a*2) and vanishes sufficiently fast when *Im*(*s*) → ±∞, then by Cauchy's theorem, the path of integration can be translated sideways inside the strip without affecting the result of the integration. More precisely, the following theorem holds:

12.3.3Theorem

If, in the strip $S(a_1, a_2)$, *F*(*s*) is holomorphic and satisfies the inequality:

$$|F(s)| \leq K|s|^{-2} \qquad (12.28)$$

for some constant *K*, then the function *f*(*t*) obtained by formula (12.27) is a continuous function of the variable $t \in (0,\infty)$ and its Mellin transform is *F*(*s*).

Remark that this result gives only a sufficient condition for the inversion formula to yield a continuous function.

From a practical point of view, it is important to note that the inversion formula applies to a function F, holomorphic in a given strip, and that the uniqueness of the result holds only with respect to that strip.

In fact, a Mellin transform consists of a pair: a function *F*(*s*) and a strip of holomorphy *S*(*a*1, *a*2). A unique function *F*(*s*) with several disjoint strips of holomorphy will in general have several reciprocals, one for each strip. Some examples will illustrate this point.

Example : The Mellin transform of the function:

$$f(t) = \left(H(t - t_0) - H(t)\right) t^z \qquad (12.29)$$

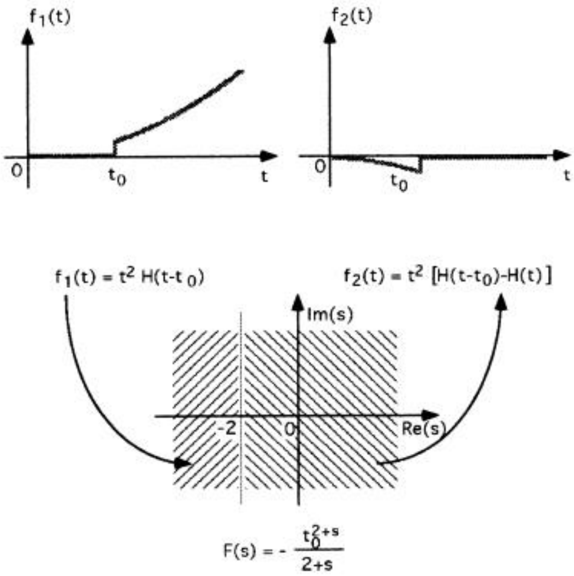

FIGURE 12.1 Examples of results when the regions of holomorphy are changed.

is given by:

$$\mathcal{M}[f; s] = -\frac{t_0^{z+s}}{(z+s)} \qquad (12.30)$$

provided $Re(s) > -Re(z)$. We see an example of two functions $F(s)$ having the same analytical expression but considered in two distinct regions of holomorphy: the inverse Mellin transforms, given respectively by (11.29) are indeed different (see **Figure 12.1**).

Corollary Let $\mathcal{M}[f;s]$ and $\mathcal{M}\ [g;s]$ be the Mellin transforms of functions *f* and *g* with strips of holomorphy *Sf* and *Sg* , respectively, and suppose that some real number c exists such that $c \in Sf$ and $1 - c \in Sg$. Then Parseval's formula can be written as:

$$\int_0^{\infty} f(t)\, g(t)\, dt = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \mathcal{M}[f;\, s]\mathcal{M}[g;\, 1-s]\, ds \qquad (12.31)$$

This formula may be established formally by computing the right-hand side of (12.31) using definition (12.15):

$$\frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \mathcal{M}[f;\, s]\mathcal{M}[g;\, 1-s]\, ds = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \mathcal{M}[g;\, 1-s] \int_0^{\infty} f(t)\, t^{s-1}\, dt\, ds \qquad (12.32)$$

Exchanging the two integrals:

$$\frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \mathcal{M}[f;\, s]\mathcal{M}[g;\, 1-s]\, ds = \frac{1}{2\pi j} \int_0^{\infty} f(t) \int_{c-j\infty}^{c+j\infty} \mathcal{M}[g;\, 1-s]\, t^{s-1}\, dt\, ds \qquad (12.33)$$

and using the inverse formula (12.27) for *g* leads to (12.31).
Different sets of conditions ensuring the validity of this Parseval formula may be stated The crucial point is the interchange of integrals that cannot always be justified.

12.4 TRANSFORMATION OF DISTRIBUTIONS

The extension of the correspondence (12.15) to distributions has to be considered to introduce a larger framework in which Dirac delta and other singular functions can be treated straightforwardly. The distributional setting of Mellin's transformation has been studied mainly by Fung Kang,10 A. H. Zemanian, and O. P. Misra and J. L. Lavoine. As we will see, several approaches of the subject are possible as it was the case for Fourier's transformation.
It is possible to define the Mellin transform for all distributions belonging to the space $\mathfrak{D}'_+$ of distributions on the half-line (0, ∞) The procedure is to start from the space $\mathfrak{D}(0,\infty)$ of infinitely differentiable functions of compact support on (0,∞) and to consider the set *Q* of their Mellin transforms.
It can be shown that it is a space of entire functions which is isomorphic, as a linear topological space, to the space *Z* of Gelfand and Shilov. This

space can be used as a space of test functions and the one-to-one correspondence thus defined between elements of spaces $\mathfrak{D}$ $(0,\infty)$ and Q can then be carried (i.e., transposed) to the dual spaces $\mathfrak{D}'_+$ and Q'. In this operation, a Mellin transform is associated with any distribution in $\mathfrak{D}'_+$ and the result belongs to a space Q' formed of analytic functionals

The situation is quite analogous to that encountered with the Fourier transformation where a correspondence between distributions spaces $\mathfrak{D}'$ and Z' is established. Actually, it may be efficient to restrict the class of distributions for which the Mellin transformation will be defined, as is usually done in Fourier analysis with the introduction of the space $\mathfrak{D}'$ of tempered distributions. In the present case, a similar approach can be based on the possibility to single out subspaces of $\mathfrak{D}'_+$ whose elements are Mellin-transformed into functions which are analytic in a given strip. This construction will now be sketched.

The most practical way to proceed is to give a new interpretation of formula (12.15) by considering it as the application of a distribution *f* to a test function *t s*–1:

$$F(s) = \langle f, t^{s-1} \rangle \qquad (12.34)$$

A suitable space of test functions $\mathcal{T}(a_1, a_2)$ containing all functions t^{s-1} for s in the region $a_1 < R_e(s)\ a_2$ may be introduced as follows. The space $\mathcal{T}(a_1, a_2)$ is composed of functions $\varphi(t)$ defined on $(0,\infty)$ and with continuous derivatives of all orders going to zero as t approaches either zero or infinity. More precisely, there exists two positive numbers ζ1,ζ2, such that, for all integers k, the following conditions hold:

$$t^{k+1-a_1-\zeta_1}\ \phi^{(k)}(t) \to 0, \quad t \to 0 \qquad (12.35)$$

$$t^{k+1-a_2-\zeta_2}\ \phi^{(k)}(t) \to 0, \quad t \to 0 \qquad (12.36)$$

A topology on $\mathcal{T}$ is defined accordingly, it can be verified that all functions in $\mathfrak{D}(0,\infty)$ belong to (a_1, a_2). The space of distributions $\mathcal{T}'(a_1, a_2)$ is then introduced as a linear space of continuous linear functional on (a_1, a_2). It may be noticed that if α_1, α2 are two real numbers such that $a1 < \alpha_1 < \alpha_2 < a_2$, then (α_1,α_2) is included in (a_1, a_2). One may so define a whole collection of ascending spaces (a_1, a_2) with compatible* topologies, thus ensuring the existence of limit spaces when $a1 \to -\infty$ and/or $a_2 \to \infty$. Hence, the dual spaces of distributions are such that $\mathcal{T}'(a1, a2) \subset \mathcal{T}'(\alpha1, \alpha2)$ and $\mathcal{T}'(-\infty, +\infty)$ is included in all of them. Moreover, as a consequence of the status of $\mathcal{T}(0,\infty)$ relatively to $\mathcal{T}(a1, a2)$, the space $\mathcal{T}'(a1, a2)$ is a subspace of distributions in $\mathfrak{D}'_+$.

With the above definitions, the Mellin transform of an element $f \in '(a1, a2)$ is defined by:

$$\mathcal{M}\left[f; s\right] \equiv F\left(s\right) = \left\langle f, t^{s-1} \right\rangle \qquad (12.37)$$

The result is always a conventional function $F(s)$ holomorphic in the strip $a1 < Re(s) < a2$.

In summary, every distribution in $\mathfrak{D}'_+$has a Mellin transform which, as a rule, is an analytic functional.

Besides, it is possible to define subspaces $\mathcal{T}'(a1, a2)$ of $\mathfrak{D}'_+$ whose elements, f, are Mellin transformed by formula (12.37) into functions $F(s)$ holomorphic in the strip $S(a1, a2)$. Any space, $\mathcal{T}'$, contains in particular Dirac distributions and arbitrary distributions of bounded support. They are stable under derivation and multiplication by a smooth function. Their complete characterization is given by the following theorems.

12.4.1Theorem (Uniqueness theorem) Let $[f;s] = F(s)$ and $[h;s] = H(s)$ be Mellin transforms with strips of holomorphy S_f and S_h, respectively. If the strips overlap and if $F(s)$ $H(s)$ for $s \in S_f \cap S_h$, then $f \equiv h$ as distributions in $\mathcal{T}'(a1, a2)$ where the interval $(a1, a2)$ is given by the intersection of $Sf \cap Sh$ with the real axis.

12.4.2 Theorem (Characterization of the Mellin transform of a distribution in $\mathcal{T}'(a_1, a_2)$. A necessary and sufficient condition for a function $F(s)$ to be the Mellin transform of a distribution $f \in'(a_1, a_2)$ is

- $F(s)$ is analytic in the strip $a1 < Re\,(s) < a2)$,
- For any closed substrip $\alpha_1 \le Re(s) \le \alpha_2$ with $a_1 < \alpha_1 < \alpha_2 < a_2$ there exists a polynomial P such that $F(s) \le P\,(s)$ for $\alpha_1 \le Re(s) \le \alpha_2$.

Check your Progress-2

3. State Inversion Formula

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4.State Characterization of the Mellin transform Theorem

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12.5 LET US SUM UP

We have discussed Fourier Transform and its properties. We discussed in detail the Mellin Transformation.

12.6 KEYWORDS

Holomorphic function: is a complex-valued function of one or more complex variables that is, at every point of its domain, complex differentiable in a neighbourhood of the point.

Expanded Form: The **expanded** notation a number is represented as the sum of each digit in a number multiplied by its place value.

12.7 QUESTIONS FOR REVIEW

1. Solve the integral equation

$$f(s) = s\int_s^{\infty} \frac{g'(t)}{(t^2 - s^2)^{\frac{1}{2}}} dt$$

2. Solve the Abel integral equation of the second kind

$$g(s) = s^{-\frac{1}{2}} e^{-\frac{a}{4s}} + \frac{i}{\sqrt{\pi}} \int_0^s g(t)/(s-t)^{\frac{1}{2}} dt$$

3. Solve the distribution by Mellin transform

$$f = \sum_{n=1}^{\infty} \delta(t - pn), \quad p > 0$$

12.8 SUGGESTED READINGS AND REFERENCES

1. M. Gelfand and S. V. Fomin. Calculus of Variations, Prentice Hall.
2. Linear Integral Equation: W.V. Lovitt (Dover).
3. Integral Equations, Porter and Stirling, Cambridge.
4. The Use of Integral Transform, I.n. Sneddon, Tata-McGrawHill, 1974
5. R. Churchil& J. Brown Fourier Series and Boundary Value Problems, McGraw-Hill, 1978
6. D. Powers, Boundary Value Problems Academic Press, 1979.

12.9 ANSWERS TO CHECK YOUR PROGRESS

1. Provide explanation and remarks– 12.2

2. Provide definition – 12.2.5

3. Provide expression with explanation – 12.3.2

4. Provide statement– 12.4.2

UNIT 13: INTEGRAL TRANSFORM METHOD III

STRUCTURE

13.0 OBJECTIVES

Understand the concept of Bessel function

Understand the Elementary properties of the Bessel functions

Understand the Hankel Transform

Comprehend the Connection with the Fourier Transform

13.1 INTRODUCTION

Notes

Hankel transforms are integral transformations whose kernels are Bessel functions. They are sometimes referred to as Bessel transforms. When we are dealing with problems that show circular symmetry, Hankel transforms may be very useful. Laplace's partial differential equation in cylindrical coordinates can be transformed into an ordinary differential equation by using the Hankel transform. Because the Hankel transform is the two-dimensional Fourier transform of a circularly symmetric function, it plays an important role in optical data processing.

13.2 DEFINITIONS AND PROPERTIES

Bessel functions are solutions of the differential equation

$$x^2 y'' + xy' + (x^2 - p^2)y = 0 \qquad (13.1)$$

where p is a parameter.

Equation (13.1) can be solved using series expansions. The Bessel function *Jp(x)* of the first kind and of order p is defined by

$$J_p(x) = \left(\frac{1}{2}x\right)^p \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{4}x^2\right)^k}{k!\,\Gamma(p+k+1)} \ . \qquad (13.2)$$

The Bessel function *Yp(x)* of the second kind and of order p is another solution that satisfies

$$W(x) = \det\begin{bmatrix} J_p(x) & Y_p(x) \\ J_p'(x) & Y_p'(x) \end{bmatrix} = \frac{2}{\pi x} \ .$$

Properties of Bessel function have been studies extensively.

13.2.1 Elementary properties of the Bessel functions are

1. Asymptotic forms.

$$J_p(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{1}{2}p\pi - \frac{1}{4}\pi\right), \qquad x \to \infty . \tag{13.3}$$

2. *Zeros.*

Jp(x) and *Yp(x)* have an infinite number of real zeros, all of which are simple, with the possible exception of $x = 0$. For nonnegative p the sth positive zero of *Jp(x)* is denoted by $j_{p,s}$. The distance between two consecutive zeros tends to π: $(j_{p,s+1} - j_{p,s}) = \pi$.

3. Integral representations.

$$J_p(x) = \frac{\left(\frac{1}{2}x\right)^p}{\pi^{1/2}\Gamma(p+1/2)} \int_0^{\pi} \cos(x\cos\theta)\sin^{2p}\theta \, d\theta \ . \tag{13.4}$$

If p is a positive integer or zero, then

$$\begin{aligned} J_p(x) &= \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - p\theta)d\theta \\ &= \frac{j^{-n}}{\pi}\int_0^{\pi} e^{jx\cos\theta}\cos(p\theta)d\theta \ . \end{aligned} \tag{13.5}$$

4. Recurrence relations.

$$J_{p-1}(x) - \frac{2p}{x} J_p(x) + J_{p+1}(x) = 0 \tag{13.6}$$

$$J_{p-1}(x) - J_{p+1}(x) = 2J'_p(x) \tag{13.7}$$

$$J'_p(x) = J_{p-1}(x) - \frac{p}{x} J_p(x) \tag{13.8}$$

$$J'_p(x) = -J_{p+1}(x) + \frac{p}{x} J_p(x) \ . \tag{13.9}$$

5. *Hankel's repeated integral*. Let $f(r)$ be an arbitrary function of the real variable r, subject to the condition that

$$\int_0^{\infty} f(r)\sqrt{r}\,dr$$

Notes

is absolutely convergent. Then for $p \geq -1/2$

$$\int_0^\infty s\,ds \int_0^\infty f(r) J_p(sr) J_p(su)\, r\, dr = \frac{1}{2}[f(u+) + f(u-)] \qquad (13.10)$$

provided that $f(r)$ satisfies certain Dirichlet condtions.

13.3 HANKEL TRANSFORM DEFINITION

Let $f(r)$ be a function defined for $r \geq 0$. The vth order Hankel transform of $f(r)$ is defined as

$$F_\nu(s) \equiv \mathcal{H}_\nu\{f(r)\} \equiv \int_0^\infty r f(r) J_\nu(sr) dr \ . \qquad (13.11)$$

If $v > -1/2$, Hankel's repeated integral immediately gives the inversion formula

$$f(r) = \mathcal{H}_\nu^{-1}\{F_\nu(s)\} \equiv \int_0^\infty s F_\nu(s) J_\nu(sr) ds \ . \qquad (13.12)$$

The most important special cases of the Hankel transform correspond to $v = 0$ and $v = 1$. Sufficient but not necessary conditions for the validity of (13.11) and (13.12) are

1. $f(r) = O(r^{-k})$, $r \rightarrow \infty$ where $k > 3/2$.
2. $f'(r)$ is piecewise continuous over each bounded subinterval of $[0, \infty)$.
3. $f(r)$ is defined as $[f(r+) + f(r-)]/2$.

These conditions can be relaxed.

13.4 CONNECTION WITH FOURIER TRANSFORM

We consider the two-dimensional Fourier transform of a function $\phi(x,y)$, which shows a circular symmetry. This means that $\phi(r\cos\theta, r\sin\theta)$ $f(\mathrm{r},\theta)$ is independent of θ.

The Fourier transform of ϕ is

$$\Phi(\zeta,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j(x\zeta,y\eta)}dx\,dy \quad . \qquad (13.13)$$

We introduce the polar coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$\zeta = s\cos\varphi, \qquad \eta = s\sin\varphi \quad .$$

We have then

$$\begin{aligned}\phi(s\cos\varphi, s\sin\varphi) \equiv F(s,\varphi) &= \frac{1}{2\pi}\int_{0}^{\infty} r\,dr\int_{0}^{2\pi} e^{-jrs\cos(\theta-\varphi)}f(r)d\theta \\ &= \frac{1}{2\pi}\int_{0}^{\infty} rf(r)dr\int_{0}^{2\pi} e^{-jrs\cos\alpha}d\alpha \\ &= \int_{0}^{\infty} rf(r)J_0(rs)dr.\end{aligned}$$

This result shows that $F(s,\phi)$ is independent of ϕ, so that we can write $F(s)$ instead of $F(s, \phi)$. Thus, the two-dimensional Fourier transform of a circularly symmetric function is, in fact, a Hankel transform of order zero.

This result can be generalized: the N-dimensional Fourier transform of a circularly symmetric functionof N variables is related to the Hankel transform of order $N/2 - 1$. If $f(r,\theta)$ depends on θ, we can expand it into a Fourier series

$$f(r,\theta) = \sum_{n=-\infty}^{\infty} f_n(r)e^{jn\theta} \qquad (13.14)$$

And similarly,

$$F(s,\varphi) = \frac{1}{2\pi}\int_0^\infty r\,dr\int_0^\infty e^{-jrs\cos(\theta-\varphi)} f(r,\theta)d\theta = \sum_{n=-\infty}^{\infty} F_n(s)e^{jn\varphi} \qquad (13.15)$$

$$f_n(r) = \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)e^{-jn\theta}d\theta \qquad (13.16)$$

where

and

$$F_n(s) = \frac{1}{2\pi}\int_0^{2\pi} F(s,\varphi)e^{-jn\varphi}d\varphi \quad . \qquad (13.17)$$

Substituting (13.15) into (13.17) and using (13.14), we obtain

$$\begin{aligned} F_n(s) &= \frac{1}{(2\pi)^2}\int_0^{2\pi} e^{-jn\varphi}d\varphi\int_0^{2\pi} d\theta\int_0^\infty f(r,\theta)e^{js\,r\cos(\theta-\varphi)}r\,dr \\ &= \frac{1}{(2\pi)^2}\int_0^{2\pi} e^{-jn\varphi}d\varphi\int_0^\infty r\,dr\int_0^{2\pi} e^{js\,r\cos(\theta-\varphi)}d\theta \times \sum_{m=-\infty}^{\infty} f_m(r)e^{jm\theta} \\ &= \frac{1}{(2\pi)}\int_0^\infty r\,dr\int_0^{2\pi} e^{-jn\alpha}e^{js\,r\cos\alpha} f_n(r)d\alpha \\ &= \int_0^\infty rf_n(r)J_n(sr)dr \\ &= \mathcal{H}_n\{f_n(r)\}. \end{aligned}$$

In a similar way, we can derive

$$f_n(r) = \mathcal{H}_n\{F_n(s)\} \quad . \qquad (13.18)$$

Check your Progress-1

1. State *Asymptotic forms of the Bessel Function*

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2.Define Hankel Transform Definition

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13.5 PROPERTIES AND EXAMPLE

Hankel transforms do not have as many elementary properties as do the Laplace or the Fourier transforms.

For example, because there is no simple addition formula for Bessel functions, the Hankel transform does not satisfy any simple convolution relation.

1. Derivatives.

Let

$$F_\nu(s) = \mathcal{H}_\nu\{f(x)\}.$$

Then

$$G_\nu(s) = \mathcal{H}_\nu\{f'(x)\} = s\left[\frac{\nu+1}{2\nu}F_{\nu-1}(s) - \frac{\nu-1}{2\nu}F_{\nu+1}(s)\right]. \qquad (13.19)$$

Proof:

$$G_\nu(s) = \int_0^\infty x f'(x)\, J_\nu(sx)\,dx$$

$$= [x f(x)\, J_\nu(sx)]_0^\infty - \int_0^\infty f(x)\frac{d}{dx}[x J_\nu(sx)]\,dx \,\cdot$$

In general, the expression between the brackets is zero, and

$$\frac{d}{dx}[x J_\nu(sx)] = \frac{sx}{2\nu}[(\nu+1)\, J_{\nu-1}(sx) - (\nu-1)\, J_{\nu+1}(sx)].$$

Hence, we have (13.19).

2. The Hankel transform of the Bessel differential operator. The Bessel differential operator

$$\Delta_\nu \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \left(\frac{\nu}{r}\right)^2 = \frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \left(\frac{\nu}{r}\right)^2$$

is derived from the Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

after separation of variables in cylindrical coordinates (r, θ, z).

Let $f(r)$ be an arbitrary function with the property that $\lim_{r\to\infty} f(r) = 0$. Then

$$\mathcal{H}_\nu\{\Delta_\nu f(r)\} = -s^2\,\mathcal{H}_\nu\{f(r)\} \quad . \qquad (13.20)$$

This result shows that the Hankel transform may be a useful tool in solving problems with cylindrical symmetry and involving the Laplacian operator.

Proof Integrating by parts, we have

$$\mathcal{H}_\nu\{\Delta_\nu f(r)\} = \int_0^\infty \left[\frac{d}{dr}r\frac{df}{dr} - \frac{\nu^2}{r}f(r)\right]J_\nu(sr)dr$$

$$= \int_0^\infty \left[s^2 J_\nu''(sr) + \frac{s}{x}J_\nu'(sr) - \frac{\nu^2}{r^2}J_\nu(sr)\right]f(r)\,r\,dr$$

$$= -s^2\int_0^\infty rf(r)J_\nu(rs)dr$$

$$= -s^2\,\mathcal{H}_\nu\{f(r)\}.$$

This property is the principal one for applications of the Hankel transforms to solving differential equations.

$$\mathcal{H}_\nu\{f(ar)\} = \frac{1}{a^2} F_\nu\left(\frac{s}{a}\right). \qquad (13.21)$$

2. Division by r.

4. Division by r.

$$\mathcal{H}_\nu\{r^{-1}f(r)\} = \frac{s}{2\nu}[F_{\nu-1}(s) + F_{\nu+1}(s)]. \qquad (13.22)$$

5.

$$\mathcal{H}_\nu\left\{r^{\nu-1}\frac{d}{dr}[r^{1-\nu}f(r)]\right\} = -sF_{\nu-1}(s). \qquad (13.23)$$

6.

$$\mathcal{H}_\nu\left\{r^{-\nu-1}\frac{d}{dr}[r^{\nu+1}f(r)]\right\} = sF_{\nu+1}(s). \qquad (13.24)$$

7. Parseval's theorem. Let

$$F_\nu(s) = \mathcal{H}_\nu\{f(r)\}$$

and

$$G_\nu(s) = \mathcal{H}_\nu\{g(r)\}.$$

Then

$$\begin{aligned}\int_0^\infty F_\nu(s)G_\nu(s)s\,ds &= \int_0^\infty F_\nu(s)s\,ds\int_0^\infty r\,g(r)J_\nu(sr)dr \\ &= \int_0^\infty r\,g(r)dr\int_0^\infty sF_\nu(s)J_\nu(sx)ds \qquad (13.25) \\ &= \int_0^\infty r\,g(r)f(r)dr.\end{aligned}$$

13.5.1 Convolution Identity

Let $f_1(r)$ and $f_2(r)$ have Hankel transforms $F1(s)$ and $F2(s)$, respectively.

$$\mathfrak{F}\left\{\int\int_{-\infty}^{\infty} f_1(\sqrt{x_1^2+y_1^2})f_2(\sqrt{(x-x_1)^2+(y-y_1)^2})dx_1dy_1\right\} = 4\pi^2 F_1(s)F_2(s).$$

We have

Hence, we have

$$\mathcal{H}_0\{f_1(r)\star\star f_2(r)\} = \frac{1}{2\pi}\mathfrak{F}_{(2)}\{f_1(r)\star\star f_2(r)\} = 2\pi F_1(s)F_2(s)$$

Therefore, to find the inverse Hankel transform of $2\pi F1(s)F2(s)$, we convolve $f_1(\sqrt{x^2+y^2})$ with $f_2(\sqrt{x^2+y^2})$ and in the answer we replace $\sqrt{x^2+y^2}$ by r. We can also write the above relationship in the form

$$\mathcal{H}_0\{2\pi f_1(r)f_2(r)\} = F_1(s)\star\star F_2(s)\ .$$

Example:

If $f_1(r) = f_2(r) = [J_1(ar)]/r$ then from the convolution identity above, we obtain

$$\mathcal{H}_0\left\{2\pi\frac{J_1^2(ar)}{r^2}\right\} = \frac{1}{a^2}p_a(s)\star\star p_a(s)$$

$$p_a(s)\star\star p_a(s) = \left(2\cos^{-1}\frac{s}{2a} - \frac{s}{a}\sqrt{1-\frac{s^2}{4a^2}}\right)a^2\ .$$

$$\mathcal{H}_0\left\{2\pi\frac{J_1^2(ar)}{r^2}\right\} = \left(2\cos^{-1}\frac{s}{2a} - \frac{s}{a}\sqrt{1-\frac{s^2}{4a^2}}\right)p_{2a}(s)$$

$$p_{2a}(s) = \begin{cases} 1 & |s| \le 2a \\ 0 & \text{otherwise}\end{cases}\ .$$

where

Hence

Example: The Hankel transform of $r^{\nu-1}e^{-ar}$, $a > 0$ is given by

$$\mathcal{H}_\nu\{r^{\nu-1}e^{-ar}\} = \int_0^\infty r^\nu e^{-ar} J_\nu(sr)dr = \frac{1}{s^{\nu+1}}\int_0^\infty t^\nu J_\nu(t) e^{-\frac{a}{s}t}dt$$

$$= \frac{1}{s^{\nu+1}}\mathcal{L}\left\{t^\nu J_\nu(t);\; p = \frac{a}{s}\right\}$$

where we set $t = rs$ and L is the Laplace transform operator .But

$$t^\nu J_\nu(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+2\nu}}{n!\Gamma(n+\nu+1)2^{2n+\nu}}$$

$$\mathcal{L}\{t^\nu J_\nu(t);\; p\} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)2^{2n+\nu}} \mathrm{L}\{t^{2n+2\nu};\; p\}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(2n+2\nu+1)}{n!\Gamma(n+\nu+1)2^{2n+\nu} p^{2n+2\nu+1}}\,.$$

The duplication formula of the gamma function gives the relationship

$$\frac{\Gamma(2n+2\nu+1)}{\Gamma(n+\nu+1)} = \frac{1}{\sqrt{\pi}} 2^{2n+2\nu}\Gamma\left(n+\nu+\frac{1}{2}\right)$$

and, therefore, the Laplace transform relation becomes

$$\mathcal{L}\{t^\nu J_\nu(t);\; p\} = \frac{2^\nu}{\sqrt{\pi} p^{2\nu+1}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma\left(n+\nu+\frac{1}{2}\right)}{n!} \left(\frac{1}{p^2}\right)^n .$$

The last series can be summed by using properties of the binomial series

$$(1+x)^{-b} = \sum_{n=0}^{\infty} \binom{-b}{n} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+b)}{n!\Gamma(b)} x^n, \qquad |x| < 1$$

where the relation

$$\binom{-b}{n} = \frac{(-1)^n b(b+1)\cdots(b+n-1)}{n!} = \frac{(-1)^n \Gamma(n+b)}{n!\Gamma(b)}$$

was used. The Laplace transform now becomes

$$\mathcal{L}\{t^{\nu} J_{\nu}(t);\ p\} = \frac{2^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}(p^2+1)^{\nu+\frac{1}{2}}} = \frac{2^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\left[\left(\frac{a}{s}\right)^2+1\right]^{\nu+\frac{1}{2}}}, \quad \mathrm{Re}(p) > 1$$

and, hence,

$$\mathcal{H}_{\nu}\{r^{\nu-1}e^{-ar}\} = \frac{1}{s^{\nu+1}} \frac{2^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\left[\left(\frac{a}{s}\right)^2+1\right]^{\nu+\frac{1}{2}}} = \frac{s^{\nu}2^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}(a^2+s^2)^{\nu+\frac{1}{2}}}, \qquad \nu > -\frac{1}{2}\ .$$

If we set $v = 0$, we obtain

$$\mathcal{H}_0\{r^{-1}e^{-ar}\} = \frac{1}{\sqrt{a^2+s^2}}, \qquad a > 0\ .$$

Example

The Hankel transform $\mathcal{H}_0\{e\ -ar\}$ is given by

$$\mathcal{H}_0\{e^{-ar}\} = \mathcal{L}\{r\ \ J_0(sr);\ \ r \to a\} = -\frac{d}{da}\left[(s^2+a^2)^{-\frac{1}{2}}\right]$$

$$= \frac{a}{[s^2+a^2]^{3/2}}, \qquad a > 0$$

since multiplication by r corresponds to differentiation in the Laplace transform domain.

Example:

We know the following identity:

$$\frac{d^2 r_n(x)}{dx^2}+\frac{1}{x}\frac{dr_n(x)}{dx}+r_n(x)=2nr_{n+1}(x)$$

where $r_n(x) = J_n(x)/x^n$.

Using the Hankel transform property of the Bessel operator, we obtain the relationship

$$(1-s^2)R_n(s)=2nR_{n+1}(s)$$

$$R_{n+1}(s)=\frac{1-s^2}{2n}R_n(s)=\cdots=\frac{(1-s^2)^n}{2^n n!}R_1(s).$$

We know,

$$\mathcal{H}_0\left\{\frac{J_1(r)}{r}\right\}=p_1(s)$$

And hence

$$\mathcal{H}_0\left\{\frac{J_n(r)}{r^n}\right\}=\frac{(1-s^2)^{n-1}}{2^{n-1}(n-1)!}p_1(s)$$

Example :

If the impulse response of a linear space invariant system is $h(r)$ and the input to the system is $f(r)$, then its output is $g(r) = f(r)**h(r)$ and, hence,

$$G(s)=2\pi F(s)H(s).$$

Since $\mathcal{H}_0\{J_0(ar)\} = [\delta(s-a)]/a$ and $\phi(s)\delta(s-a) = \phi(a)\delta(s-a)$, we conclude that if the input is $f(r) = J_0(ar)$, then

$$G(s)=\frac{2\pi}{a}\delta(s-a)H(s)=\frac{2\pi H(a)}{a}\delta(s-a).$$

Therefore, the output is

$$g(r) = 2\pi H(a)\, J_0(ar)\ .$$

Check your Progress-2

3. Explain *The Hankel transform of the Bessel differential operator*

4.State Convolution Identity

13.6 LET US SUM UP

Hankel transform is the two-dimensional Fourier transform of a circularly symmetric function, it plays an important role in optical data processing.

13.7 KEYWORDS

Identity : an **identity** is an equality relating one **mathematical** expression A to another **mathematical** expression B, such that A and B (which might contain some variables) produce the same value for all values of the variables within a certain range of validity

Domain. The **domain** of a **function** is the complete set of possible values of the independent variable.

13.8 QUESTIONS FOR REVIEW

1. State the Elementary properties of the Bessel functions

3. Explain Connection with fourier Transform

13.9 SUGGESTED READINGS AND REFERENCES

7. M. Gelfand and S. V. Fomin. Calculus of Variations, Prentice Hall.
8. Linear Integral Equation: W.V. Lovitt (Dover).
9. Integral Equations, Porter and Stirling, Cambridge.
10. The Use of Integral Transform, I.n. Sneddon, Tata-McGrawHill, 1974
11. R. Churchil& J. Brown Fourier Series and Boundary Value Problems, McGraw-Hill, 1978
12. D. Powers, Boundary Value Problems Academic Press, 1979.

13.10 ANSWERS TO CHECK YOUR PROGRESS

1. Provide the property– 13.2.1

2. Provide definition – 13.3

3. Provide statement and proof – 13.5 [2^{nd} point]

4. Provide identity– 13.5.1

UNIT 14: INTEGRAL TRANSFORM AND BOUNDARY VALUE APPLICATION

STRUCTURE

14.0 OBJECTIVES

Understand the boundary value application of Hankel Transform

Understand the The Finite Hankel Transform

14.1 INTRODUCTION

The Hankel transform is an integral transform and was first developed by the mathematician Hermann Hankel. It is also known as the Fourier–Bessel transform. Just as the Fourier transform for an infinite interval is related to the Fourier series over a finite interval, so the Hankel transform over an infinite interval is related to the Fourier–Bessel series over a finite interval.

14.2 THE ELECTRIFIED DISC

Let υ be the electric potential due to a flat circular electrified disc, with radius R = 1, the center of the disc being at the origin of the three-dimensional space and its axis along the z-axis. In polar coordinates, the potential satisfies Laplace's equation

$$\nabla^2 \upsilon \equiv \frac{\partial^2 \upsilon}{\partial r^2} + \frac{1}{r}\frac{\partial \upsilon}{\partial r} + \frac{\partial^2 \upsilon}{\partial z^2} = 0 . \qquad (14.1)$$

The boundary conditions are

$$\upsilon(r,0) = \upsilon_0, \qquad 0 \le r < 1 \qquad (14.2)$$

$$\frac{\partial \upsilon}{\partial z}(r,0) = 0, \qquad r > 1 . \qquad (14.3)$$

In (14.2), υ0 is the potential of the disc. Condition (14.3) arises from the symmetry about the plane z = 0.

Let

$$V(s,z) = \mathcal{H}_0\{\upsilon(r,z)\}$$

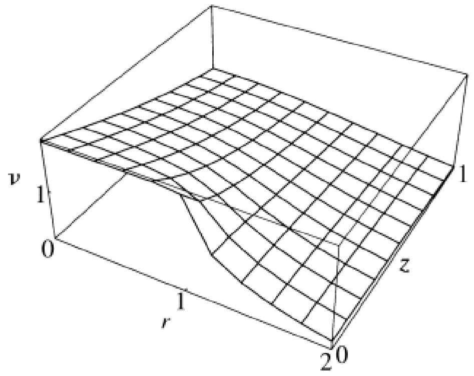

FIGURE (14.1)Electrical potential due to an electrified disc.

so that

$$\mathcal{H}_0\{\nabla^2 v\} = -s^2 V(s,z) + \frac{\partial^2 V}{\partial z^2}(s,z) = 0 \ .$$

The solution of this differential equation is

$$V(s,z) = A(s)e^{-sz} + B(s)e^{sz}$$

where *A* and *B* are functions that we have to determine using the boundary conditions.

$$v(r,z) = \int_0^\infty sA(s)e^{-sz}J_0(sr)ds \ . \qquad (14.3)$$

Because the potential vanishes as *z* tends to infinity, we have *B*(*s*) 0. By inverting the Hankel transform, we have

The boundary conditions are now

$$v(r,0) = \int_0^\infty sA(s)J_0(rs)ds = v_0, \qquad 0 \le r < 1 \qquad (14.4)$$

$$\frac{\partial v}{\partial z}(r,0) = \int_0^\infty s^2 A(s)J_0(rs)ds = 0, \qquad r > 1 \ . \qquad (14.5)$$

we see that *A*(*s*) = sin *s/s* 2 so that

$$\upsilon(r,z) = \frac{2\upsilon_0}{\pi}\int_0^{\infty} \frac{\sin s}{s} e^{-sz} J_0(sr)ds \; . \qquad (14.6)$$

In Figure 14.1, the graphical representation of υ(r,z) for υ0 = 1 is depicted on the domain $0 \le r \le 2$, $0 \le z \le 1$. The evaluation of υ(*r*,*z*) requires numerical integration.

Equations (14.4) and (14.5) are special cases of the more general pair of equations

$$\int_0^{\infty} f(t)t^{2\alpha} J_\nu(xt)dt = a(x), \qquad 0 \le x < 1 \qquad (14.7)$$

$$\int_0^{\infty} f(t) J_\nu(xt)dt = 0, \qquad x > 1 \qquad (14.8)$$

where *a*(*x*) is given and *f*(*x*) is to be determined.
The solution of (14.3) can be expressed as a repeated integral

If *a*(*x*) = *x* β, and α < 1, 2α + β > –3/2, α + *v* > –1, *v* > –1, then

$$f(x) = \frac{2^{-\alpha} x^{1-\alpha}}{\Gamma(\alpha+1)}\int_0^1 s^{-\nu-\alpha} J_{\nu+\alpha}(xs)\frac{d}{ds}\int_0^s a(t)t^{\nu+1}(s^2-t^2)^{\alpha}\,dt\,ds, \quad -1<\alpha<0 \qquad (14.9)$$

$$f(x) = \frac{(2x)^{1-\alpha}}{\Gamma(\alpha)}\int_0^1 s^{-\nu-\alpha+1} J_{\nu+\alpha}(xs)\int_0^s a(t)t^{\nu+1}(s^2-t^2)^{\alpha-1}\,dt\,ds, \quad 0<\alpha<1 \; . \qquad (14.10)$$

$$f(x) = \frac{\Gamma\left(1+\frac{\beta+\nu}{2}\right)x^{-(2\alpha+\beta+1)}}{2^{\alpha}\Gamma\left(1+\alpha+\frac{\beta+\nu}{2}\right)}\int_0^x t^{\alpha+\beta+1} J_{\nu+\alpha}(t)dt \; . \qquad (14.11)$$

With β = *v* and α < 1, α + *v* > – 1, *v* > – 1 further simplification is possible:

$$f(x)=\frac{\Gamma(\nu+1)}{(2x)^{\alpha}\Gamma(\nu+\alpha+1)}\,J_{\nu+\alpha+1}(x)\ . \qquad (14.12)$$

14.3HEAT CONDUCTION

Heat is supplied at a constant rate *Q* per unit area and per unit time through a circular disc of radius *a* in the plane *z* = 0, to the semi-infinite space *z* > 0. The thermal conductivity of the space is *K*. The plane *z* = 0 outside the disc is insulated. The temperature is denoted by υ(*r*,*z*). We have again the Laplace Equation (14.1) in polar coordinates, but the boundary conditions are now

$$\begin{aligned} -K\frac{\partial \upsilon(r,z)}{\partial z} &= Q, \qquad r<a,\ \ z=0 \\ &= 0, \qquad r>a,\ \ z=0\ . \end{aligned} \qquad (14.13)$$

The Hankel transform of the differential equation is again

$$\frac{\partial^2 V}{\partial z^2}(s,z)-s^2V(s,z)=0\,. \qquad (14.14)$$

We can now transform also the boundary condition,

$$-K\frac{\partial V}{\partial z}(s,0)=Qa\ \ J_1(as)/\,s\ . \qquad (14.15)$$

The solution of (14.13) must remain finite as *z* tends to infinity. We have

$$V(s,z)=A(s)e^{-sz}\ .$$

Using condition (14.15) we can determine

$$A(s)=Qa\ \ J_1(as)/(Ks^2)\ .$$

Consequently, the temperature is given by

$$\upsilon(r,z) = \frac{Qa}{K}\int_0^\infty e^{-sz} J_1(as) J_0(rs) s^{-1} ds \, . \qquad (14.16)$$

14.4 THE LAPLACE EQUATION IN THE HALFSPACE Z > 0, WITH A CIRCULARLY SYMMETRIC DIRICHLET CONDITION AT Z = 0

We try to find the solution υ(*r*,*z*) of the boundary value problem

$$\begin{cases} \dfrac{\partial^2 \upsilon}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \upsilon}{\partial r} + \dfrac{\partial^2 \upsilon}{\partial z^2} = 0, & z > 0, \ 0 < r < \infty \\ \upsilon(r,0) = f(r). \end{cases} \qquad (14.17)$$

Taking the Hankel transform of order $\mathcal{H}_0$ yields

$$\frac{\partial^2 V}{\partial z^2}(s,z) - s^2 V(s,z) = 0$$

and

$$V(s,0) = \int_0^\infty r f(r)\, J_0(sr)\, dr \, .$$

The solution is

$$V(s,z) = e^{-sz}\int_0^\infty r f(r)\, J_0(sr)\, dr$$

so that

$$\upsilon(r,z) = \int_0^\infty s\, e^{-sz} J_0(sr) ds \int_0^\infty p\, f(p) J_0(sp) dp \, . \qquad (14.18)$$

For the special case

$$f(r) = h(a - r)$$

where $h(r)$ is the unit step function, we have the solution

$$\upsilon(r, z) = a\int_0^\infty e^{-sz} J_0(sr) J_1(as) ds \,. \qquad (14.19)$$

Check your Progress-1

1. Explain *the Heat Conduction application of Hankel's Transform*

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2.Discuss The Laplace Equation in the Halfspace z > 0, with a Circularly Symmetric Dirichlet Condition at z = 0

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14.5 AN ELECTROSTATIC PROBLEM

The electrostatic potential $Q(r,z)$ generated in the space between two grounded horizontal plates at $z = \pm l$ by a point charge q at $r = 0$, $z = 0$ shows a singular behavior at the origin. It is given by

$$\upsilon(r, z) = \varphi(r, z) + q(r^2 + z^2)^{-1/2} \qquad (14.20)$$

where $\phi(r,z)$ satisfies Laplace's Equation (9.26). The boundary conditions are

$$\varphi(r, \pm\ell) + q(r^2 + \ell^2)^{-1/2} = 0 \,. \qquad (14.21)$$

Taking the Hankel transform of order 0, we obtain

$$\frac{\partial^2\Phi}{\partial z^2}(s, z) - s^2\Phi(s, z) = 0 \qquad (14.22)$$

$$\Phi(s, \pm\ell) = -\frac{qe^{-s\ell}}{s} \qquad (14.23)$$

The solution is

$$A(s)e^{-sz} + B(s)e^{sz}$$

where $A(s)$ and $B(s)$ must satisfy

$$A(s)e^{+s\ell} + B(s)e^{-s\ell} = -\frac{q\ e^{-s\ell}}{s}$$

$$A(s)e^{-s\ell} + B(s)e^{s\ell} = -\frac{q\ e^{-s\ell}}{s}\ .$$

Hence

$$A(s) = B(s) = -\frac{q\ e^{-s\ell}}{2s\cosh(s\ell)}$$

and

$$\Phi(s, z) = -\frac{q\ e^{-s\ell}}{s}\frac{\cosh(sz)}{\cosh(s\ell)}\ .$$

Hence

$$\varphi(r, z) = \frac{q}{\sqrt{r^2 + z^2}} - q\int_0^\infty e^{-s\ell}\frac{\cosh(sz)}{\cosh(s\ell)}J_0(sr)\,ds\ . \qquad (14.24)$$

14.6 THE FINITE HANKEL TRANSFORM

We consider the integral transformation

$$F_\nu(\alpha) = H_\nu\{f,\alpha\} = \int_0^1 rf(r)\, J_\nu(\alpha r)\, dr \;. \qquad (14.25)$$

A property of this transformation is that

$$H_\nu(\Delta_\nu f,\alpha) = -\alpha^2 F_\nu(\alpha) + [\, J_\nu(\alpha)\, f'(1) - \alpha J'_\nu(\alpha)\, f(1)]$$

where

v is the Bessel differential operator.

If α is equal to the *s*th positive zero *jv,s of* Jv(*x*), we have

$$H_\nu(\Delta_\nu f, j_{\nu,s}) = -j^2_{\nu,s} H_\nu(f, j_{\nu,s}) + j_{\nu,s} J_{\nu+1}(j_{\nu,s}) f(1) \;.$$

If α is equal to the *s*th positive root β*v*,*s* of

$$hJ_\nu(x) + xJ'_\nu(x) = 0$$

where *h* is a nonnegative constant, we have

$$H_\nu(\Delta_\nu f, \beta_{\nu,s}) = -\beta^2_{\nu,s} H_\nu(f, \beta_{\nu,s}) + J_\nu(\beta_{\nu,s})[hf(1) + f'(1)] \;.$$

The transformation (14.25) with α = *j v,s, s* = 1, 2, … is the finite Hankel transform. It maps the function

f(*r*) into the vector (*Fv*(*jv*,1), *Fv*(*jv*, 2), *Fv*(*j v*,3) …). The inversion formula can be obtained from the wellknown theory of Fourier-Bessel series

$$f(r) = 2\sum_{s=1}^{\infty} \frac{F_\nu(j_{\nu,s})}{J^2_{\nu+1}(j_{\nu,s})}\, J_\nu(j_{\nu,s} r) \;. \qquad (14.26)$$

The transformation (14.25) with α = β*v* ,*s*, *s* = 1, 2, … is the modified finite Hankel transform. The inversion

$$f(r) = 2\sum_{s=1}^{\infty} \frac{\beta^2_{\nu,s} F_\nu(\beta_{\nu,s})}{h^2 + \beta^2_{\nu,s} - \nu^2}\, \frac{J_\nu(\beta_{\nu,s} r)}{J^2_\nu(\beta_{\nu,s})} \;. \qquad (14.27)$$

Formula is

If $h = 0$, $\beta v,s$ is the sth positive zero of denoted by $j'_{v,s}$

14.6.1 Application

We calculate the temperature $\upsilon(r,t)$ at time t of a long solid cylinder of unit radius. The initial temperature is unity and radiation takes place at the surface into the surrounding medium maintained at zero temperature.

The mathematical model of this problem is the diffusion equation in polar coordinates

$$\frac{\partial^2 \upsilon}{\partial r^2} + \frac{1}{r}\frac{\partial \upsilon}{\partial r} = \frac{\partial \upsilon}{\partial t}, \qquad 0 \le r < 1, \ \ t > 0 \qquad (14.28)$$

The initial condition is

$$\upsilon(r,0) = 1, \qquad 0 \le r \le 1 \ . \qquad (14.29)$$

The radiation at the surface of the cylinder is described by the mixed boundary condition

$$\frac{\partial \upsilon}{\partial r}(1,t) = -h\upsilon(1,t) \qquad (14.30)$$

where h is a positive constant. Transformation of (14.28) by the modified finite Hankel transform yields

$$\frac{dV}{dt}(\beta_{0,s}, t) = -\beta_{0,s}^2 V(\beta_{0,s}, t) \qquad (14.31)$$

$$V(\alpha, t) = \int_0^1 r\upsilon(r,t) J_0(\alpha r)dr$$

where

$$V(\alpha, 0) = \int_0^1 rJ_0(\alpha r)dr = \frac{J_1(\alpha)}{\alpha} \ . \qquad (14.33)$$

so that

The solution of (14.32), with the initial condition (14.33), is

$$V(\beta_{0,s}, t) = \frac{J_1(\beta_{0,s})}{\beta_{0,s}} e^{-\beta_{0,s}^2 t}.$$

Using the inversion formula, we obtain

$$\upsilon(r,t) = 2\sum_{j=1}^{\infty} e^{-\beta_{0,s}^2 t} \frac{\beta_{0,s} J_1(\beta_{0,s})}{h^2 + \beta_{0,s}^2} \frac{J_0(\beta_{0,s} r)}{J_0^2(\beta_{0,s})}. \quad (14.34)$$

14.7 RELATED TRANSFORM

For some applications, Hankel transforms with a more general kernel may be useful. We give one example.

We consider the cylinder function

$$Z_\nu(s,r) = J_\nu(sr)Y_\nu(s) - Y_\nu(sr)J_\nu(s). \quad (14.35)$$

Using this function as a kernel, we can construct the following transform pair:

$$F_\nu(s) = \int_1^{\infty} rf(r) Z_\nu(s,r)\, dr \quad (14.36)$$

$$f(r) = \int_0^{\infty} sF_\nu(s) \frac{Z_\nu(s,r)}{J_\nu^2(s) + Y_\nu^2(s)}\, ds \quad (14.37)$$

The inversion formula follows immediately from Weber's integral theorem

$$\int_1^\infty u\ du \int_0^\infty f(s) Z_\nu(r,u) Z_\nu(s,u) s\ ds = \frac{1}{2}[J_\nu^2(r) + Y_\nu^2(r)][f(r+) + f(r-)]\ . \qquad (14.38)$$

For this reason, we will refer to (9.61) and (9.62) as the Weber transform. This transform has the following important property:

If

$$f(x) = g''(x) + \frac{1}{x} g'(x) - \frac{\nu^2}{x^2} g(x) \qquad (14.39)$$

$$F_\nu(s) = -s^2 G_\nu(s) - \frac{2}{\pi} g(1)\ . \qquad (14.40)$$

We may expect that this transform is useful for solving Laplace's equation in cylindrical coordinates, with a boundary condition at $r = 1$.

Example

We want to compute the steady-state temperature $u(r,z)$ in a horizontal infinite homogeneous slab of thickness 2, through which there is a vertical circular hole of radius 1. The horizontal faces are held at temperature zero and the circular surface in the hole is at temperature T_0. The mathematical model is

$$\begin{aligned} &\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0 \\ &u(r,\ell) = u(r,-\ell) = 0 \\ &u(1,z) = T_0\ . \end{aligned} \qquad (14.41)$$

Taking the Weber transform of order zero, we have

$$\frac{\partial^2 U_0}{\partial z^2}(s,z) - s^2 U_0(s,z) = \frac{2}{\pi} T_0\ .$$

The solution of this ordinary differential equation, satisfying the boundary condition, is

$$U_0(s,z) = \frac{2T_0}{\pi s^2}\left[\frac{\cosh sz}{\cosh s\ell} - 1\right].$$

Notes

Consequently, we have

$$u(r,z)=\frac{2T_0}{\pi}\int_0^{\infty}\frac{1}{s}\left[\frac{\cosh sz}{\cosh s\ell}-1\right]\frac{Z_0(s,r)}{J_0^2(s)+Y_0^2(s)}ds \qquad (r>1). \qquad (14.42)$$

14.7.1 Need of Numerical Integration Methods

When using the Hankel transform for solving partial differential equations, the solution is found as an integral of the form

$$I(a,\ p,\ \nu)=\int_0^a J_\nu(px)f(x)dx \qquad (14.43)$$

where *a* is a positive real number or infinite. In most cases, analytical integration of (14.43) is impossible,and numerical integration is necessary. But integrals of type (14.43) are difficult to evaluate numerically if

1. The product *ap* is large.
2. *a* is infinite.
3. *f* (*x*) shows a singular or oscillatory behavior.

In cases 1 and 2, the difficulties arise from the oscillatory behavior of *Jv*(*x*) and they grow when the oscillations become stronger.
We give here a survey of numerical methods that are especially suited for the evaluation of *I* (*a, p, v*) when *ap* is large or *a* is infinite. We restrict ourselves to cases where *f*(*x*) is smooth, or where *f*(*x*) = *x* α*g* (*x*) where *g* (*x*) is smooth and α is a real number.

Check your Progress-2

3. State and discuss application of the *Finite Hankel Transform*

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4.Explain Related Transform

14.7.2 Tables of Hankel Transforms

Table 9.1 lists the Hankel transform of some particular functions for the important special case $v = 0$.

Table 9.2 lists Hankel transforms of general order v. In these tables, h(x) is the unit step function, *Iv* and *Kv* are modified Bessel functions, **L**0 and **H**0 are Struve functions, and Ker and Kei are Kelvin functions as defined in Abramowitz and Stegun.

TABLE 9.1 Hankel Transforms of Order 0.

	$f(r)$	$F_0(s) = \mathcal{H}_0\{f(r)\}$
(1)	$\frac{1}{r}$	$\frac{1}{s}$
(2)	$r^{-\mu}, \quad 1/2 < \mu < 2$	$2^{1-\mu}\frac{\Gamma(1-\frac{\mu}{2})}{\Gamma(\frac{\mu}{2})}\frac{1}{s^{2-\mu}}$
(3)	$h(a-r)$	$\frac{a}{s}J_1(as)$
(4)	e^{-ar}	$\frac{a}{(s^2+a^2)^{3/2}}$
(5)	$\frac{e^{-ar}}{r}$	$\frac{1}{\sqrt{s^2+a^2}}$
(6)	$\frac{1-e^{-ar}}{r^2}$	$\log\left(\frac{a+\sqrt{a^2+s^2}}{s}\right)$
(7)	$\log\left(1+\frac{a^2}{r^2}\right)$	$\frac{2}{s}\left[\frac{1}{s} - a\,K_1(as)\right]$
(8)	$\frac{\sin r}{r}$	$\frac{1}{\sqrt{1-s^2}}, \quad s < 1$ $0, \quad s > 1$
(9)	$\frac{\sin r}{r^2}$	$\frac{\pi}{2}, \quad s \le 1$ $\arcsin\frac{1}{s}, \quad s > 1$
(10)	$\frac{\sin(ar)}{r^2+b^2}$	$\frac{\pi}{2}e^{-ab}I_0(bs), \quad 0 < s < a$

(11)	$\dfrac{\cos(ar)}{r^2+b^2}$	$\cosh(ab)K_0(bs),\ a<s<\infty$
(12)	$e^{-a^2r^2}$	$\dfrac{e^{-s^2/4a^2}}{2a^2}$
(13)	$\dfrac{1}{r(r+a)}$	$\dfrac{\pi}{2}[\mathbf{H}_0(as)-Y_0(as)]$
(14)	$\dfrac{1}{r^2+a^2}$	$K_0(as)$
(15)	$\dfrac{1}{r(r^2+a^2)}$	$\dfrac{\pi}{2a}[I_0(as)-\mathbf{L}_0(as)]$
(16)	$\dfrac{1}{1+r^4}$	$-\,\mathrm{Kei}(s)$
(17)	$\dfrac{r^3}{1+r^4}$	$\mathrm{Ker}(s)$
(18)	$\dfrac{1}{\sqrt{r^2+a^2}}$	$\dfrac{e^{-sa}}{s}$
(19)	$\dfrac{1}{\sqrt{r^4+a^4}}$	$K_0(as/\sqrt{2})\,J_0(as/\sqrt{2})$
(20)	$\dfrac{1-J_0(ar)}{r^2}$	$\log\dfrac{a}{s},\quad s\le a$ $0,\quad s\ge a$
(21)	$\dfrac{a}{r}J_1(ar)$	$1,\quad \text{if } 0<s<a$ $0,\quad \text{if } s>a$
(22)	$\dfrac{1}{r}J_0(2\sqrt{ar})$	$\dfrac{1}{s}J_0\left(\dfrac{a}{s}\right)$

TABLE 9.2 Hankel Transforms of General Order ν.

	$f(r)$	$F_\nu(s)=\mathcal{H}_\nu\{f(r)\}$
(1)	$\dfrac{1}{r}$	$\dfrac{1}{s}$
(2)	$r^{-\mu},\ \dfrac{1}{2}<\mu<\nu+2$	$\dfrac{2^{1-\mu}}{s^{2-\mu}}\dfrac{\Gamma(\frac{\nu+2-\mu}{2})}{\Gamma(\frac{\nu+\mu}{2})}$
(3)	$x^\nu(a^2-r^2)^\mu h(a-r),$ $\mu>-1$	$2^\mu a^{\mu+\nu+1}s^{-\mu-1}\Gamma(\mu+1)J_{\nu+\mu+1}(as)$
(4)	$\dfrac{\sin ar}{r}$	$\dfrac{1}{(s^2-a^2)^{1/2}}\sin\left(\nu\arcsin\left(\dfrac{a}{s}\right)\right)\quad s>a$ $\cos\left(\dfrac{n\nu}{2}\right)\dfrac{1}{(a^2-s^2)^{1/2}}\dfrac{s^\nu}{(a+(a^2-s^2)^{1/2})^\nu}\quad s<a$
(5)	$\dfrac{\sin ar}{r^2}$	$\dfrac{\nu^{-1}s^\nu}{\left(a+\sqrt{a^2-s^2}\right)^\nu}\sin\dfrac{\nu\pi}{2}\quad s\le a$ $\nu^{-1}\sin\left(\nu\arcsin\left(\dfrac{a}{s}\right)\right)\quad s>a$
(6)	$\dfrac{e^{-ar}}{r}$	$\dfrac{(\sqrt{s^2+a^2}-a)^\nu}{s^\nu\sqrt{s^2+a^2}}$
(7)	$\dfrac{e^{-ar}}{r^2}$	$\dfrac{(\sqrt{s^2+a^2}-a)^\nu}{\nu s^\nu}$
(8)	$r^{\nu-1}e^{-ar}$	$\dfrac{(2s)^\nu\Gamma(\nu+1/2)}{(s^2+a^2)^{\nu+1/2}\sqrt{\pi}}$

(9)	$r^{\nu}e^{-ar}$	$\dfrac{2a(2s)^{\nu}\Gamma(\nu+3/2)}{(s^2+a^2)^{\nu+3/2}\sqrt{\pi}}$
(10)	$e^{-a^2r^2}r^{\nu}$	$\dfrac{s^{\nu}}{(2a^2)^{\nu+1}}\exp\left(-\dfrac{s^2}{4a^2}\right)$
(11)	$e^{-a^2r^2}r^{\mu}$	$\dfrac{\Gamma((\nu+\mu+2)/2)\left(\frac{1}{2}\frac{s}{a}\right)^{\nu}}{2a^{\mu+2}\Gamma(\nu+1)} \times {}_1F_1\left(\dfrac{\nu+\mu+2}{2};\nu+1;-\dfrac{s^2}{4a^2}\right)$
(12)	$\dfrac{r^{\nu}}{(r^2+a^2)^{\mu+1}}$	$\dfrac{s^{\mu}a^{\nu-\mu}}{2^{\mu}\Gamma(\mu+1)}K_{\nu-\mu}(as)$
(13)	$\dfrac{r^{\nu}}{(r^4+4a^4)^{\nu+\frac{1}{2}}}$	$\dfrac{\left(\frac{1}{2}s\right)^{\nu}\sqrt{\pi}}{(2a)^{2\nu}\Gamma\left(\nu+\frac{1}{2}\right)}J_{\nu}(as)K_{\nu}(as)$
(14)	$\dfrac{r^{\nu+2}}{(r^4+4a^4)^{\nu+\frac{1}{2}}}$	$\dfrac{\left(\frac{1}{2}s\right)^{\nu}\sqrt{\pi}}{2(2a)^{2\nu-2}\Gamma\left(\nu+\frac{1}{2}\right)}J_{\nu-1}(as)K_{\nu-1}(as)$
(15)	$r^{\mu-\nu}J_{\mu}(ar)$	$0 \qquad 0<s<a$ $\dfrac{2^{\mu-\nu+1}a^{\mu}(s^2-a^2)^{\nu-\mu-1}}{s^{\nu}\Gamma(\nu-\mu)} \qquad a<s$

14.8 LET US SUM UP

We have discussed the boundary value application of the Hankel Transform. We also discussed about The Finite Hankel Transform and its application. We have seen some related transform and Hankel Tables.

14.9 KEYWORDS

Transform :A **transformation** is a process that manipulates a polygon or other two-dimensional object on a plane or coordinate system

Unit step function: **functions** whose values change abruptly at specified values of time t

14.10 QUESTIONS FOR REVIEW

1. Discuss any two applications of the Hankel Transform in detail

2. Explain The Finite Hankel Transform

14.11 SUGGESTED READINGS AND REFERENCES

13. M. Gelfand and S. V. Fomin. Calculus of Variations, Prentice Hall.
14. Linear Integral Equation: W.V. Lovitt (Dover).
15. Integral Equations, Porter and Stirling, Cambridge.
16. The Use of Integral Transform, I.n. Sneddon, Tata-McGrawHill, 1974
17. R. Churchil& J. Brown Fourier Series and Boundary Value Problems, McGraw-Hill, 1978
18. D. Powers, Boundary Value Problems Academic Press, 1979.

14.12 ANSWERS TO CHECK YOUR PROGRESS

1. Provide explanation– 14.3
2. Provide explanation – 14.4
3. Provide explanation – 14.6.1
4. Provide explanation – 14.7